GRASP: BASIC COMPONENTS AND ENHANCEMENTS

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Abstract. GRASP (Greedy Randomized Adaptive Search Procedures) is a multistart metaheuristic for producing good-quality solutions of combinatorial optimization problems. Each GRASP iteration is usually made up of a construction phase, where a feasible solution is constructed, and a local search phase which starts at the constructed solution and applies iterative improvement until a locally optimal solution is found. While, in general, the construction phase of GRASP is a randomized greedy algorithm, other types of construction procedures have been proposed. Repeated applications of a construction procedure yields diverse starting solutions for the local search. This chapter gives an overview of GRASP describing its basic components and enhancements to the basic procedure, including reactive GRASP and intensification strategies.

1. Introduction

Combinatorial optimization problems involve a finite number of alternatives: given a finite solution set X and a real-valued objective function $f : X \to R$, one seeks a solution $x^* \in X$ with $f(x^*) \leq f(x)$, $\forall\ x \in X$. Several combinatorial optimization problems can be solved in polynomial time, but many of them are computationally intractable since exact polynomial-time algorithms are unknown [59]. To find an optimal solution it is theoretically possible to enumerate all solutions and evaluate each with respect to the stated objective function f. Optimal seeking methods that do not explicitly require an examination of each alternative have been developed in the last decades, such as Branch & Bound, Cutting Planes, and Dynamic Programming. Nevertheless, most real-world problems found in industry and government are either computationally intractable by their nature, or sufficiently large so as to preclude the use of exact algorithms. In such cases, heuristic methods are usually employed to find good, but not necessarily guaranteed optimal solutions. The effectiveness of these methods depends upon their ability to adapt to a particular realization, avoid entrapment at local optima, and exploit the basic structure of the problem. Building on these notions, various heuristic search techniques have been developed that have demonstrably improved our ability to obtain good solutions to difficult combinatorial optimization problems. The most promising of such techniques include simulated annealing [79], tabu search [61, 62, 65], genetic algorithms [70], variable neighborhood search [72], and GRASP (Greedy Randomized Adaptive Search Procedures) [46, 47].

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A GRASP is a multi-start or iterative process [88]. Each GRASP iteration is usually made up of a construction phase, where a feasible solution is constructed, and a local search phase which starts at the constructed solution and applies iterative improvement until a locally optimal solution is found. Repeated applications of the construction procedure yields diverse starting solutions for the local search and the best overall solution is kept as the result.

This chapter gives an overview of GRASP describing its basic components and demonstrates, step by step, how to develop such heuristics for combinatorial optimization problems. Alternative construction mechanisms and local search characteristics are described in Subsections 2.1 and 2.2, respectively. Enhancements to the basic procedure, including reactive GRASP and intensification strategies, are discussed in Section 3. Section 4 reports a number of GRASP implementations that have appeared in the literature, covering a wide range of applications. Finally, concluding remarks are given in the last section.

2. A basic GRASP

Given a finite solution set X and a real-valued objective function $f : X \rightarrow R$ to be minimized, a basic GRASP metaheuristic [46, 47] is a multi-start or iterative method, in which each iteration consists of two phases: construction of a solution and local search.

```
procedure GRASP(f(·), g(·), MaxIterations, Seed)
1    x_best:=∅; f(x_best):=+∞;
2    for k = 1, 2, ...,MaxIterations→
3        x:=ConstructGreedyRandomizedSolution(Seed, g(·));
4        if (x not feasible) then
5            x:=repair(x);
6        endif
7        x:=LocalSearch(x, f(·));
8        if (f(x) < f(x_best)) then
9            x_best:=x;
10       endif
11   endfor;
12   return(x_best);
end GRASP
```

Figure 1. Pseudo-code of a basic GRASP for a minimization problem.

The construction phase builds a solution x. If x is not feasible, a repair procedure is invoked to obtain feasibility. Once a feasible solution x is obtained, its neighborhood is investigated by the local search until a local minimum is found. The best overall solution is kept as the result. An extensive survey of the literature is given in [55]. The pseudo-code in Figure 1 illustrates the main blocks of a GRASP procedure for minimization, in which `MaxIterations` iterations are performed and `Seed` is used as the initial seed for the pseudorandom number generator.

Starting from an empty solution, a complete solution is iteratively constructed in the construction phase, one element at a time (see Figure 2). The basic GRASP construction phase is similar to the semi-greedy heuristic proposed independently

```
procedure ConstructGreedyRandomizedSolution(Seed, g(·))
1    x:=∅;
2    Sort the candidate elements i according to their incremental
     costs g(i);
3    while (x is not a complete solution)→
4        RCL:=MakeRCL();
5        v:=SelectIndex(RCL, Seed);
6        x := x ∪ {v};
7        Resort remaining candidate elements j according to their
         incremental costs g(j);
8    endwhile;
9    return(x);
end ConstructGreedyRandomizedSolution;
```

FIGURE 2. Basic GRASP construction phase pseudo-code.

by [74]. At each construction iteration, the choice of the next element to be added is determined by ordering all candidate elements (i.e. those that can be added to the solution) in a candidate list C with respect to a greedy function $g : C \rightarrow R$. This function measures the (myopic) benefit of selecting each element. The heuristic is adaptive because the benefits associated with every element are updated at each iteration of the construction phase to reflect the changes brought on by the selection of the previous element. The probabilistic component of a GRASP is characterized by randomly choosing one of the best candidates in the list, but not necessarily the top candidate. The list of best candidates is called the *restricted candidate list* (RCL). This choice technique allows for different solutions to be obtained at each GRASP iteration, but does not necessarily compromise the power of the adaptive greedy component of the method.

```
procedure LocalSearch(x, f(·))
1    Let N(x) be the neighborhood of x;
2    H:={y ∈ N(x) | f(y) < f(x)};
3    while (|H| > 0)→
4        x:=Select(H);
5        H:={y ∈ N(x) | f(y) < f(x)};
6    endwhile
7    return(x);
end GRASP
```

FIGURE 3. Pseudo-code of a generic local search procedure.

As is the case for many deterministic methods, the solutions generated by a GRASP construction are not guaranteed to be locally optimal with respect to simple neighborhood definitions. Hence, it is almost always beneficial to apply a local search to attempt to improve each constructed solution. A local search algorithm works in an iterative fashion by successively replacing the current solution by a better solution in the neighborhood of the current solution. It terminates when no better solution is found in the neighborhood. The *neighborhood structure* N

for a problem relates a solution s of the problem to a subset of solutions $N(s)$. A solution s is said to be *locally optimal* if in $N(s)$ there is no better solution in terms of objective function value. The key to success for a local search algorithm consists of the suitable choice of a neighborhood structure, efficient neighborhood search techniques, and the starting solution. Figure 3 illustrates the pseudo-code of a generic local search procedure for a minimization problem.

While such local optimization procedures can require exponential time [76] from an arbitrary starting point, empirically their efficiency significantly improves as the initial solution improves. The result is that often many GRASP solutions are generated in the same amount of time required for the local optimization procedure to converge from a single random start. Furthermore, the best of these GRASP solutions is generally significantly better than the single solution obtained from a random starting point.

It is difficult to formally analyze the quality of solution values found by using the GRASP methodology. However, there is an intuitive justification that views GRASP as a repetitive sampling technique. Each GRASP iteration produces a sample solution from an unknown distribution of all obtainable results. The mean and variance of the distribution are functions of the restrictive nature of the candidate list. For example, if the cardinality of the restricted candidate list is limited to one, then only one solution will be produced and the variance of the distribution will be zero. Given an effective greedy function, the mean solution value in this case should be good, but probably suboptimal. If a less restrictive cardinality limit is imposed, many different solutions will be produced implying a larger variance. Since the greedy function is more compromised in this case, the mean solution value should degrade. Intuitively, however, by order statistics and the fact that the samples are randomly produced, the best value found should outperform the mean value. Indeed, often the best solutions sampled are optimal.

An especially appealing characteristic of GRASP is the ease with which it can be implemented. Few parameters need to be set and tuned, and therefore development can focus on implementing efficient data structures to assure quick GRASP iterations. Finally, GRASP can be trivially implemented in parallel. Each processor can be initialized with its own copy of the procedure, the instance data, and an independent random number sequence. The GRASP iterations are then performed in parallel with only a single global variable required to store the best solution found over all processors.

2.1. **Construction mechamisms.** The construction phase is usually an iterative procedure. As underlined in Section 2, in the basic GRASP, at each construction iteration, the choice of the next element to be added to the partial solution (initially empty) is determined by ordering all candidate elements in a candidate list C with respect to their incremental costs given by evaluating a greedy function $g : C \rightarrow R$. The RCL then is the list of best candidates. The heuristic is adaptive because the incremental costs associated with every element are updated at each iteration of the construction phase to reflect the changes brought on by the selection of the previous element. The probabilistic component of a GRASP is characterized by randomly choosing one element from the RCL, but not necessarily the top candidate.

In the literature, other construction methods have been proposed. For example, in the *sample greedy* construction, instead of randomizing the greedy algorithm, a greedy algorithm is applied to each solution in a random sample of candidates. At

```
procedure ConstructGreedyRandomizedSolution(Seed, α, k, g(·))
1    x:=∅;
2    Initialize the candidate set C by all elements;
3    Evaluate the incremental cost g(i) for all i ∈ C;
4    while (|C| > 0)→
5        g_min := min_{i∈C} g(i); g_max := max_{i∈C} g(i);
6        if (CB RCL is used) then
7            Sort candidate elements i ∈ C according to their
             incremental costs g(i);
8            RCL:=C[1 ··· k];
9        else RCL:={i ∈ C | g(i) ≤ g_min + α(g_max − g_min)};
10       endif;
11       v:=SelectIndex(RCL, Seed);
12       x := x ∪ {v};
13       Update the candidate set C;
14       Reevaluate the incremental costs g(i) for all i ∈ C;
15   endwhile;
16   return(x);
end ConstructGreedyRandomizedSolution;
```

FIGURE 4. Refined pseudo-code of the GRASP construction phase.

each step, a fixed-size subset of the candidates is sampled and the incremental contribution to the cost of the partial solution is computed for each sampled element. An element with the best incremental contribution is selected and added to the partial solution. In *random plus greedy* construction, a partial random solution is built and a greedy algorithm is applied to complete the construction.

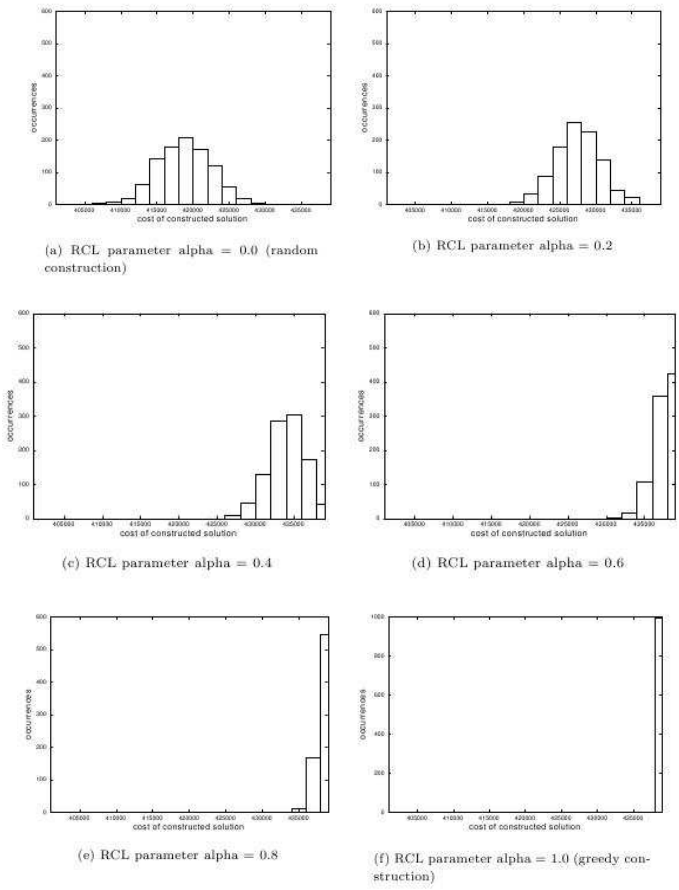
In the following, we will first explain various ways to build the RCL and then we will describe enhancements and alternative techniques for the construction phase of GRASP. Since Mockus et al. [97] pointed out that GRASP with a fixed nonzero RCL parameter α is not asymptotically convergent to a global optimum[1], several remedies have been proposed to get around this problem. They include Reactive GRASP, cost perturbations in place of randomized selection, bias functions, memory and learning, and local search on partially constructed solutions.

Construction of the RCL. Without loss of generality, let us consider a minimization problem as formulated in Section 1. At any GRASP iteration, let $g(i)$ be the incremental cost associated with the incorporation of element i in the solution under construction and let g_{min} and g_{max} be the smallest and the largest incremental costs, respectively, i.e.

$$g_{min} = \min_{i \in C} g(i), \qquad g_{max} = \max_{i \in C} g(i).$$

The restricted candidate list RCL is made up of elements $i \in C$ with the best (i.e., the smallest) incremental costs $g(i)$. There are two main mechamisms to build this list: a *cardinality-based* (CB) and a *value-based* (VB) mechanism. In the CB case, the RCL is made up of the k elements with the best incremental costs, where k is a parameter. In the VB case, the RCL is associated with a parameter $\alpha \in [0, 1]$ and a

[1]During construction, a fixed RCL parameter may rule out a candidate that is present in all optimal solutions



(a) RCL parameter alpha = 0.0 (random construction)

(b) RCL parameter alpha = 0.2

(c) RCL parameter alpha = 0.4

(d) RCL parameter alpha = 0.6

(e) RCL parameter alpha = 0.8

(f) RCL parameter alpha = 1.0 (greedy construction)

FIGURE 5. Distribution of construction phase solution values as a function of the RCL parameter α (1000 repetitions were recorded for each value of α).

threshold value $\mu = g_{min} + \alpha(g_{max} - g_{min})$. In fact, all candidate elements i whose incremental cost $g(i)$ is no greater than the threshold value are inserted into the RCL, i.e. $g(i) \in [g_{min}, \mu]$. Note that, the case $\alpha = 0$ corresponds to a pure greedy algorithm, while $\alpha = 1$ is equivalent to a random construction. The pseudo-code in Figure 4 is a refinement of the greedy randomized contruction pseudo-code shown in Figure 2.

Each GRASP construction procedure produces a sample solution from an unknown distribution, whose mean and variance strongly depends on the mechanism used to build the RCL. If the RCL has only one element, then the same solution will be produced in all iterations. The variance of the distribution will be zero and the mean will be equal to the value of the greedy solution. Instead, when the RCL contains more elements, many different solutions will be produced, implying a larger variance. Since greediness plays a smaller role in this case, the mean solution value should be worse. However, the value of the best solution found outperforms the mean value and very often is optimal. In Figure 5 we report through

six histograms the results of a study conducted by Resende and Ribeiro [116]. The histograms refer to an instance of MAXSAT with 100 variables and 850 clauses and depicts results obtained with 1000 independent constructions using the VB mechanisms of the GRASP construction procedure described in [113, 114]. Since this is a maximization problem, the purely greedy construction corresponds to $\alpha = 1$, whereas the random construction occurs with $\alpha = 0$. Note that, when the value of α increases from 0 to 1, the mean solution value increases towards the purely greedy solution value, while the variance approaches zero.

Prais and Ribeiro in [107, 108] observed the behavior of GRASP and the quality of the GRASP output solutions for different RCL construction mechanisms, based on different strategies for the variation of the value of the parameter α:

(a) α is randomly chosen from a uniform discrete probability distribution;
(b) α is randomly chosen from a decreasing non-uniform discrete probability distribution;
(c) fixed value of α, close to the purely greedy choice.

The authors incorporated these three strategies into the GRASP procedures developed for four different optimization problems: (1) matrix decomposition for traffic assignment in communication satellite [109]; (2) set covering [46]; (3) weighted MAX-SAT [113, 114]; and (4) graph planarization [115, 119]. The resulting heuristics have been tested on a subset of state-of-the-art instances for each type of problem. The total number of iterations performed was fixed at 10,000. The observed conclusions can be summized as follows. Strategy (c) presented the shortest average computation times for three out the four problem types. It was also the one with the least variability in the constructed solutions and, in consequence, found the best solution the fewest times. Strategy (a) presented a high number of hits and this behavior also illustrates the effectiveness of strategies based on the variation of the RCL parameter.

In [107, 109], the authors also tested GRASP with a further RCL construction mechanism, in which the parameter α is self-tuned and its value is periodically modified according to the quality of the obtained solutions. This extension of the basic GRASP is called *Reactive GRASP* and will be described in detail in the next paragraph.

Reactive GRASP. The results of the study conducted in [107, 109] involving variation of the value of the RCL parameter α motivated the proposal of the extension of the basic GRASP called Reactive GRASP. Prais and Ribeiro in [109] have shown that using a single fixed value for the value of RCL parameter α very often hinders finding a high-quality solution, which eventually could be found if another value was used. Moreover, one drawback of the basic GRASP is the lack of *learning* from the history of solutions found in previous iterations. The basic algorithm discards information about any solution encountered that does not improve the incumbent. Instead, it is worth to use information gathered from good solutions leading to *memory-based* procedures. Information about the quality of proviously generated solutions can influence the construction phase, by modifying the selection probabilities associated with each element of the RCL.

In this paragraph, we describe Reactive GRASP, the first enhancement that incorporates a learning mechanism in the memoryless construction phase of the basic GRASP. In Reactive GRASP, the value of the RCL parameter α is selected in each iteration from a discrete set of possible values with a probability that depends

on the solution values found along the previous iterations. One way to accomplish this is to use the rule proposed in [109]. Let $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be the set of possible values for α. At the first GRASP iteration, all m values have the same probability to be selected, i.e.

$$p_i = \frac{1}{m}, \qquad i = 1, 2, \ldots, m.$$

At any subsequent iteration, let $\hat{z}$ be the incumbent solution and let A_i be the average value of all solutions found using $\alpha = \alpha_i$, $i = 1, \ldots, m$. The selection probabilities are periodically reevaluated as follows:

$$p_i = \frac{q_i}{\sum_{j=1}^{m} q_j},$$

where $q_i = \hat{z}/A_i$, $i = 1, \ldots, m$. If values of $\alpha = \alpha_i$ ($i \in \{1, \ldots, m\}$) lead to the best solutions on average, then the value of q_i is increased and larger values of q_i correspond to more suitable values for α. The probabilities associated with these more appropriate values will then increase when they are reevaluated.

Due to greater diversification and less reliance on parameter tuning, Reactive GRASP has lead to improvements over the basic GRASP in terms of robustness and solution quality. In fact, it has been successfully applied in power system transmission network planning [29] and in a capacitated location problem [41].

Cost perturbations. Another step toward an improved and alternative solution construction mechanism is to allow *cost perturbations*. The idea is to introduce some "noise" in the original costs in a fashion that resembles the noising method of Charon and Hudry [36, 37]. Cost perturbations are effective in all cases when the construction algorithm is not very sensitive to randomization, as for example in the case of the Steiner problem in graphs. To solve this problem, the hybrid GRASP procedure proposed by Ribeiro et al. in [122] used as one of the main building blocks of the construction phase the shortest-path heuristic of Takahashi and Matsuyama [131].

Another situation where cost perturbations can be effective is when there is no greedy algorithm available for the problem to be solved, as for example in the case of the prize-collecting Steiner tree problem. To solve this problem, the hybrid GRASP procedure proposed by Canuto et al. in [33] used the primal-dual algorithm of Goemans and Williamson [69] to build initial solutions using perturbed costs. More especifically, in [33], at each iteration a new solution for the prize-collecting Steiner tree problem is built using node prizes updated by a perturbation function, according to the structure of the current solution. Two different prize perturbation schemes are used to enforce search diversification, as described in the following.

***Perturbation by eliminations*::** The primal-dual algorithm used in the construction phase is driven to build a new solution without some of the nodes appearing in the solution constructed in the previous iteration. This is done by changing to zero the prizes of some persistent nodes, which appeared in the last solution built and remained at the end of the local search. A parameter α controls the fraction of the persistent nodes whose prizes are temporarily set to zero;

***Perturbation by prize changes*::** Similarly to the noising method of Charon and Hudry [36, 37], some noise is introduced into the node prizes, resulting

in a change of the objective function as well. For each node i, a perturbation factor $\beta(i)$ is randomly generated in the interval $[1-a, 1+a]$, where a is an implementation parameter. The original prize $\pi(i)$ associated with node i is temporarily changed to $\pi(i) = \pi(i) \cdot \beta(i)$.

Experimental results have shown that embedding a strategy of costs perturbation into a GRASP framework improves the best overall results. The hybrid GRASP with path-relinking proposed for the Steiner problem in graphs by Ribeiro et al. in [122] uses this cost perturbation strategy and is among the most effective heuristics currently available. Path-relinking will be described in detail in Subsection 3.1.

Bias functions. Another construction mechanism was proposed by Bresina [32]. Once built the RCL, instead of choosing with equal probability one candidate among the RCL elements, Bresina introduced a family of probability distributions to bias the selection toward some particular candidates. A bias function is based on a rank $r(x)$ assigned to each candidate x according to its greedy function value and is evaluated only for the elements in RCL. Several different bias functions were introduced:

i. random bias: $\text{bias}(r(x)) = 1$;
ii. linear bias: $\text{bias}(r(x)) = 1/r(x)$;
iii. log bias: $\text{bias}(r(x)) = \log^{-1}[r(x)+1]$;
iv. exponential bias: $\text{bias}(r(x)) = e^{-r}$;
v. polynomial bias of order n: $\text{bias}(r(x)) = r^{-n}$.

Let $\text{bias}(r(x))$ be one of the bias functions defined above. Once these values have been evaluated for all elements of the RCL, the probability p_x of selecting element x is

$$p_x = \frac{\text{bias}(r(x))}{\sum_{y \in RCL} \text{bias}(r(y))}.$$

A successful application of Bresina's bias function can be found in [28], where experimental results show that the evaluation of bias functions may be restricted only to the elements of the RCL.

Reactive GRASP has been the first and very simple attempt to enhance the basic GRASP in order to keep trace and use history from previous iterations. Another very simple attempt is due to Fleurent and Glover [58] who proposed improved constructive multistart strategies that besides defining a special bias function also maintains a pool of *elite solutions* to be used in the construction phase. To become an elite solution, a solution must be either better than the best member of the pool, or better than its worst member and suffciently different from the other solutions in the pool, in order to preserve not only solution quality but also the diversity of solutions. Fleurent and Glover defined: 1) a *strongly determined variable* as one that cannot be changed without eroding the objective or changing significantly other variables; 2) a *consistent variable* as one that receives a particular value in a large portion of the elite solution set, and 3) for each solution component i, a measure $I(i)$ of its strongly determined and consistent features that becomes larger as i appears more often in the pool of elite solutions. The intensity function $I(i)$ is used in the construction phase as follows. Recall that $g(i)$ is the greedy function, i.e. the incremental cost associated with the insertion of element i into the solution under construction. Let $K(i) = F(g(i), I(i))$ be a function of the greedy and the intensification functions. The idea of Fleurent and Glover is to define a special bias

function that depends on $K(\cdot)$. In fact, the intensification scheme biases selection from the RCL to those elements i with a high value of $K(i)$ by setting its selection probability to be

$$p_i = \frac{K(i)}{\sum_{y \in RCL} K(y)}.$$

They suggested $K(i) = \lambda g(i) + I(i)$, with $K(i)$ varying with time by changing the value of λ, e.g. initially λ may be set to a large value that is decreased when diversification is called for. Rules and procedures for changing the value of λ are given by Fleurent and Glover [58] and Binato et al. [28].

POP in construction. The intuitive idea behind the *Proximate Optimality Principle* (POP) is that "good solutions at one level are likely to be found 'close' to good solutions at an adjacent level". Fleurent and Glover [58] proposed a GRASP for the quadratic assignment problem that applies local search not only at the end of each construction phase, but also during the construction itself. the scope of this further application of local search is to "iron-out" from the current solution its "bad" components. Neverthelss, experimental investigation conducted in the literature has shown that applying the POP idea at each construction iteration is excessively running time consuming. One possibility to implement the idea in a more efficient way is to apply local search during a few points in the construction phase and not during each construction iteration. In Binato et al. [28], local search is applied after 40% and 80% of the construction moves have been taken, as well as at the end of the construction phase.

2.2. **Local search.** In this section, we focus on local search design strategies. As underlined in Section 2, the solutions generated by a GRASP construction are not guaranteed to be locally optimal with respect to simple neighborhood definitions. Hence, it is almost always beneficial to apply a local search to attempt to improve each constructed solution. A local search algorithm works in an iterative fashion by successively replacing the current solution by a better solution in the neighborhood of the current solution. It terminates when no better solution is found in the neighborhood. The *neighborhood structure* N for a given problem relates a solution s of the problem to a subset of solutions $N(s)$. A solution s is said to be *locally optimal* if in $N(s)$ there is no better solution in terms of objective function value.

Let us consider, for example, the MAX-CUT problem. Given an undirected graph $G = (V, E)$, where $V = \{1, \ldots, n\}$ is the set of vertices and E is the set of edges, and weights w_{ij} associated with the edges $(i, j) \in E$, the MAX-CUT problem consists in finding a subset of vertices S such that the weight of the cut $(S, \bar{S})$ given by

$$w(S, \bar{S}) = \sum_{i \in S, j \in \bar{S}} w_{ij}$$

is maximized. A feasible solution is a n-binary vector x, such that

$$x_i = \begin{cases} 1, & \text{if } i \in S; \\ 0, & \text{otherwise.} \end{cases}$$

Generally speaking, given a feasible solution $x \in X$, the elements of the neighborhood $N(x)$ of x are those solutions that can be obtained by applying to x an elementary modification, called *move*. In the case of MAX-CUT, for example, let

us consider a simple graph having only three nodes and let $x = (1, 0, 0)$ be a feasible solution. Then, the so called 1-flip neighborhood of x (in general, any feasible solution that can be represented by a binary vector) is the set of all binary arrays that differ from x by exactly one element. Therefore, if $x = (1, 0, 0)$, then $N(x) = \{(0, 0, 0), (1, 1, 0), (1, 0, 1)\}$.

Formally, a local search starts from an initial solution $x_0 \in X$ and iteratively generates a sequence of improving solutions $x_1, \ldots, x_M$, where $M =$ `MaxIterations`. At the k-th iteration, $k = 1, \ldots, M$, x_k is locally optimal respect to the neighborhood $N(x_{k-1})$ since $N(x_{k-1})$ is searched for an improving solution x_k such that $f(x_k) < f(x_{k-1})$. If such a solution is found, it is made the current solution. Otherwise, the search ends with x_{k-1} as a local optimum.

The effectiveness of local search depends on several factors, such as the neighborhood structure, the function to be minimized, and the starting solution. A solution x is said to be in the *basin of attraction* of the global optimum if local search starting from x leads to the global optimum. Once the neighborhood and objective function are determined, different starting solutions can be used to start the local search in a multi-start procedure.

Randomly generated solutions are of poor quality on average. Even if a randomly generated solution is in the basin of attraction of a good quality local optimal solution, the number of moves needed to reach the local optimum can be large, even exponential in the problem size [76]. The greedy algorithm usually produces solutions of better quality than those of randomly generated solutions. Furthermore, using greedy solutions as starting points for local search in a multi-start procedure will usually lead to good, though, most often, suboptimal solutions. This is because the amount of variability in greedy solutions is small and it is less likely that a greedy starting solution will be in the basin of attraction of a global optimum. A greedy randomized construction as the one embedded in GRASP adds variability to the greedy algorithm.

In [116], besides analyzing the quality of the solution obtained by varying between randomness and greediness in the VB mechanisms of the GRASP construction procedure, Resende and Ribeiro also analyzed the quality of the solutions output of the local search starting from solutions obtained by applying VB mechanisms with different values for the α parameter. In Figure 6 we report the results of the study conducted by Resende and Ribeiro through six histograms.

The histograms refer to the same instance of MAXSAT with 100 variables and 850 clauses and depict results obtained by applying local search to each of the 1000 constructed solutions. Overall, Resende and Ribeiro observed that the the variance of the overall solution diversity, final solution quality, and running time increased with the variance of the solution values obtained in the construction phase. Moreover, it is unlikely that GRASP will find an optimal solution if the average solution value is low, even if there is a large variance in the overall solution values, such as is the case for $\alpha = 0$. On the other hand, if there is little variance in the overall solution values, it is also unlikely that GRASP will find an optimal solution, even if the average solution is high, as is the case for $\alpha = 0$. Good solutions are usually obtained in the presence of relatively high average solution values and of a relatively large variance, such as is the case for $\alpha = 0.8$.

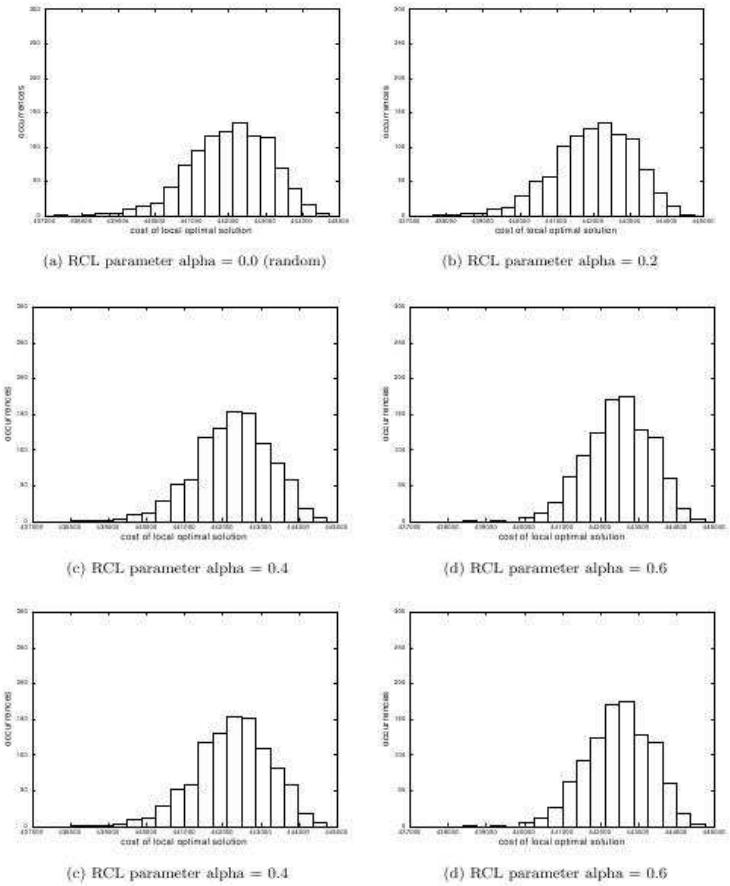


FIGURE 6. Distribution of of local search phase solution values as a function of the RCL parameter α (1000 repetitions were recorded for each value of α).

3. Enhancements

A number of enhancements to the basic GRASP, presented in the previous Sections, have been proposed in the literature. They include long-term memory, the proximate optimality principle, and bias functions in a GRASP.

In this section, we review the use of path-relinking.

3.1. **Path-relinking.** Path-relinking is an enhancement to the basic GRASP procedure, leading to significant improvements in solution quality. Path-relinking was originally proposed by Glover [63] as an intensification strategy exploring trajectories connecting elite solutions obtained by tabu search or scatter search [64, 66, 67]. It can be traced back to the pioneering work of Kernighan and Lin [78]. Starting from one or more elite solutions, paths in the solution space leading towards other guiding elite solutions are generated and explored in the search for better solutions. This is accomplished by selecting moves that introduce attributes contained in the guiding solutions. At each iteration, all moves that incorporate attributes

of the guiding solution are analyzed and the move that best improves (or least deteriorates) the initial solution is chosen.

The first proposal of a hybrid GRASP with path-relinking was in 1999 due to Laguna and Martí [85]. It was followed by several extensions, improvements, and successful applications [4, 33, 52, 54].

Path-relinking is applied to a pair of solutions $\mathbf{x}$ and $\mathbf{y}$, where one can be the solution obtained from the current GRASP iteration, and the other is a solution from an elite set of solutions. $\mathbf{x}$ is called the *initial solution* and $\mathbf{y}$ the *guiding solution*. The set $\mathcal{E}$ of elite solutions has usually a fixed size that does not exceed `MaxElite`. Given the pair $\mathbf{x}, \mathbf{y}$, their common elements are kept constant, and the space of solutions spanned by these elements is searched with the objective of finding a better solution. The size of the solution space grows exponentially with the the distance between the *initial* and *guiding* solutions and therefore only a small part of the space is explored by path-relinking. The procedure starts by computing the symmetric difference $\Delta(\mathbf{x}, \mathbf{y})$ between the two solutions, i.e. the set of moves needed to reach $\mathbf{y}$ (target solution) from $\mathbf{x}$ (initial solution). A path of solutions is generated linking $\mathbf{x}$ and $\mathbf{y}$. The best solution x^* in this path is returned by the algorithm. Since there is no guarantee that x^* is locally optimal, often local search is applied, starting from x^*, and the resulting locally optimal solution is returned.

Let us denote the set of solutions spanned by the common elements of the n-vectors $\mathbf{x}$ and $\mathbf{y}$ as

$$S(\mathbf{x}, \mathbf{y}) := \{w \text{ feasible} \mid w_i = x_i = y_i, i \notin \Delta(\mathbf{x}, \mathbf{y})\} \setminus \{\mathbf{x}, \mathbf{y}\}. \tag{1}$$

Clearly, $|S(\mathbf{x}, \mathbf{y})| = 2^{n-d(\mathbf{x},\mathbf{y})} - 2$, where $d(\mathbf{x}, \mathbf{y}) = |\Delta(\mathbf{x}, \mathbf{y})|$. The underlying assumption of path-relinking is that there exist good-quality solutions in $S(\mathbf{x}, \mathbf{y})$, since this space consists of all solutions which contain the common elements of two good solutions $\mathbf{x}$ and $\mathbf{y}$. Since the size of this space is exponentially large, a greedy search is usually performed where a path of solutions

$$\mathbf{x} = \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{d(\mathbf{x},\mathbf{y})}, \mathbf{x}_{d(\mathbf{x},\mathbf{y})+1} = \mathbf{y},$$

is built, such that $d(\mathbf{x}_i, \mathbf{x}_{i+1}) = 1, \;\; i = 0, \ldots, d(\mathbf{x}, \mathbf{y})$, and the best solution from this path is chosen. Note that, since both $\mathbf{x}$ and $\mathbf{y}$ are, by construction, local optima in some neighborhood $N(\cdot)$[2], then in order for $S(\mathbf{x}, \mathbf{y})$ to contain solutions which are not contained in the neighborhoods of $\mathbf{x}$ or $\mathbf{y}$, $\mathbf{x}$ and $\mathbf{y}$ must be sufficiently distant from each other.

Figure 7 illustrates the pseudo-code of the path-relinking procedure applied to the pair of solutions $\mathbf{x}$ (starting solution) and $\mathbf{y}$ (target solution). In line 1, an initial solution $\mathbf{x}$ is select at random among the elite set elements and usually it differs sufficiently from the guiding solution $\mathbf{y}$. The loop in lines 6 through 14 computes a path of solutions $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{d(\mathbf{x},\mathbf{y})-2}$, local search is applied in line 15, and the solution x^* with the best objective function value is returned in line 16. This is achieved by advancing one solution at a time in a greedy manner. At each iteration, the procedure examines all moves $m \in \Delta(x, \mathbf{y})$ from the current solution x and selects the one which results in the least cost solution (line 7), i.e. the one which minimizes $f(x \oplus m)$, where $x \oplus m$ is the solution resulting from applying move m to solution x. The best move m^* is made, producing solution $x \oplus m^*$ (line 9). The set of available moves is updated (line 8). If necessary, the best solution x^*

[2]Where the same metric $d(\mathbf{x}, \mathbf{y})$ is usually used.

```
procedure Path-relinking(f(·), x, E)
1     Choose, at random, a pool solution y ∈ E to relink with x;
2     Compute symmetric difference Δ(x, y);
3     f* := min{f(x), f(y)};
4     x* := arg min{f(x), f(y)};
5     x := x;
6     while (Δ(x, y) ≠ ∅) →
7           m* := arg min{f(x ⊕ m) | m ∈ Δ(x, y)};
8           Δ(x ⊕ m*, y) := Δ(x, y) \ {m*};
9           x := x ⊕ m*;
10          if (f(x) < f*) then
11                f* := f(x);
12                x* := x;
13          endif;
14    endwhile;
15    x* := LocalSearch(x*, f(·));
16    return (x*);
end Path-relinking;
```

FIGURE 7. Pseudo-code of a generic path-relinking for a minimization problem.

is updated (lines 10–13). The procedure terminates when $\mathbf{y}$ is reached, i.e. when $\Delta(x, \mathbf{y}) = \emptyset$, returning the best solution found.

We now describe a possible way to hybridize with path-relinking the basic GRASP described in Section 2. The integration of the path-relinking procedure with the basic GRASP is shown in Figure 8. The pool $\mathcal{E}$ of elite solutions is initially empty, and until it reaches its maximum size no path relinking takes place. After a solution $\mathbf{x}$ is found by GRASP, it is passed to the path-relinking procedure to generate another solution. The procedure `AddToElite`$(\mathcal{E}, x_p)$ attempts to add to the elite set of solutions the solution that was just found. Since we wish to maintain a pool of good but diverse solutions, each solution obtained by path-relinking is considered as a candidate to be inserted into the pool if it is sufficiently different from every other solution currently in the pool. If the pool already has `MaxElite` solutions and the candidate is better than the worst of them, then a simple strategy is to have the former replace the latter. Another strategy, which tends to increase the diversity of the pool, is to replace the pool element most similar to the candidate among all pool elements with cost worse than the candidate's.

More formally, in several papers, a solution x_p is added to the elite set $\mathcal{E}$ if either one of the following conditions holds:

(1) $f(x_p) < \min\{f(\mathbf{w}) : \mathbf{w} \in \mathcal{E}\}$,
(2) $\min\{f(\mathbf{w}) : \mathbf{w} \in \mathcal{E}\} \leq f(x_p) < \max\{f(\mathbf{w}) : \mathbf{w} \in \mathcal{E}\}$ and $d(x_p, \mathbf{w}) > \beta n, \ \forall \mathbf{w} \in \mathcal{E}$, where β is a parameter between 0 and 1 and n is the number of decision variables.

If x_p satisfies either of the above, it then replaces an elite solution $\mathbf{z}$ no better than x_p and most similar to x_p, i.e. $\mathbf{z} = \text{argmin}\{d(x_p, \mathbf{w}) : \mathbf{w} \in \mathcal{E} \text{ such that } f(\mathbf{w}) \geq f(x_p)\}$.

```
procedure GRASP+PR(f(·), g(·), MaxIterations, Seed, MaxElite)
1    x_best:=∅; f(x_best):=+∞; E := ∅
2    for k = 1, 2, ...,MaxIterations→
3        x:=ConstructGreedyRandomizedSolution(Seed, g(·));
4        if (x not feasible) then
5            x:=repair(x);
6        endif
7        x:=LocalSearch(x, f(·));
8        if (k ≤MaxElite) then
9            E := E ∪ {x};
10           if (f(x) < f(x_best)) then
11               x_best:=x;
12           endif
13       else
14           x_p:=Path-relinking(f(·), x, E);
15           AddToElite(E, x_p);
16           if (f(x_p) < f(x_best)) then
17               x_best:=x_p;
18           endif
19       endif
20   endfor;
21   return(x_best);
end GRASP
```

FIGURE 8. Pseudo-code of a basic GRASP with path-relinking heuristic for a minimization problem.

Figure 8 shows the simplest way to combine GRASP with path-relinking, which is applied as an intensification strategy to each local optimum obtained after the GRASP local search phase.

More generally, two basic strategies can be used:

i. path-relinking is applied as a post-optimization step to all pairs of elite solutions;
ii. path-relinking is applied as an intensification strategy to each local optimum obtained after the local search phase.

Applying path-relinking as an intensification strategy to each local optimum (strategy ii.) seems to be more effective than simply using it as a post-optimization step [117].

3.2. **Parallel GRASP.** Parallel implementation of GRASP is very straightforward. Most parallel implementations of GRASP follow the *multiple-walk independent thread* strategy, based on the distribution of the iterations over the processors [10, 11, 48, 94, 101, 102]. Two strategies have been at most proposed:

1) *Search space decomposition*, where the search space is partitioned into several regions and GRASP is applied to each in parallel.
2) *Iteration parallelization*, where the GRASP iterations are partitioned and each partition is assigned to a processor.

In general, each search thread has to perform `MaxIterations`$/p$ iterations, where p is the number of processors. Each processor has a copy of the sequential algorithm, a copy of the problem data, and an independent seed to generate its own pseudo-random number sequence. To avoid that the processors find the same solutions, each of them must use a different sequence of pseudorandom numbers. A single global variable is required to store the best solution found over all processors. One of the processors acts as the master, reading and distributing problem data, generating the seeds which will be used by the pseudorandom number generators at each processor, distributing the iterations, and collecting the best solution found by each processor. Since the iterations are completely independent and very little information is exchanged, linear speedups are easily obtained provided that no major load imbalance problems occur. The iterations may be evenly distributed over the processors or according with their demands, to improve load balancing.

The efficiency of multiple-walk independent-thread parallel implementations of metaheuristics, based on running multiple copies of the same sequential algorithm, has been also addressed for a number of metaheuristics. A given target value f_t for the objective function f is broadcast to all processors which independently execute the sequential algorithm. All processors halt immediately after one of them finds a solution x whose objective function value $f(x)$ is at least as good as f_t. The speedup is given by the ratio between the times needed to find x, using respectively the sequential algorithm and the parallel implementation with p processors. For a number of metaheuristics, these speedups are linear, such as for example simulated annealing [43, 99] and tabu search, provided that the search starts from a local optimum [27, 130].

Aiex et al. [7] have shown experimentally that the solution times for GRASP also have this property, showing that they fit a two-parameter exponential distribution. The same result holds approximately when GRASP is implemented in conjunction with path-relinking [4]. In the case of the multiple-walk independent thread implementation described in [4] for the 3-index assignment problem, each processor applies path-relinking to pairs of elite solutions stored in a local pool. Computational results using MPI on an SGI Challenge computer with 28 R10000 processors showed linear speedups.

Cooperative-thread strategies may be also implemented using path-relinking, by combining elite solutions stored in a central pool with the local optima found by each processor at the end of each GRASP iteration. Canuto et al. [33] used path-relinking to implement a parallel GRASP for the prize-collecting Steiner tree problem. In their strategy, pairs of elite solutions from a centralized unique central pool are distributed to the processors which perform path-relinking in parallel.

For more detail about independent and cooperative parallelizations of GRASP, the reader can refer to [6, 118].

3.3. **GRASP in hybrid metaheuristics.** The combination of the basic GRASP with path-relinking, discussed in Subsection 3.1, is one among the first attempts made to embed GRASP in hybrid metaheuristic schemes. Nevertheless, as enhancements to its basic framework, different hybridizations of GRASP with several other metaheuristics have been studied and proposed in the literature. In this section, some of them are surveyed and briefly described.

Laguna and Gonzalez-Velarde in 1991 [84] have first studied hybridization of GRASP with tabu search. Later, in 1999 Delmaire et al. [41] proposed a Reactive

GRASP whose local search have been strengthened by tabu search. In particular, they have proposed two approaches. In the first, GRASP is applied as a powerful diversification strategy in the context of a tabu search procedure. The second approach is an implementation of the Reactive GRASP algorithm, in which the local search phase is strengthened by tabu search. Results reported for the capacitated location problem show that the hybrid approaches perform better than the pure methods previously used.

GRASP has been used also in conjunction with genetic algorithms. Basically, the feasible solution found by using a GRASP construction phase has been used as initial population by a genetic algorithm, as for example in [20] and in [3], where a greedy genetic algorithm is proposed for the quadratic assignment problem.

Another interesting hybridization of GRASP involves VNS (Variable Neighborhood Search) and Variable Neighborhood Descent (VND) proposed by Hansen and Mladenović [73, 96]. Almost all randomization effort in the basic GRASP algorithm involves the construction phase, while local search stops at the first local optimum. On the other hand, strategies such as VNS and VND rely almost entirely on the randomization of the local search to escape from local optima. With respect to this issue, probabilistic strategies such as GRASP and VNS may be considered as complementary and potentially capable of leading to effective hybrid methods.

VNS is based on the exploration of a dynamic neighborhood model. Contrary to other metaheuristics based on local search methods, VNS allows changes of the neighborhood structure along the search. It explores increasingly distant neighborhoods of the current best found solution x. Each step has three major phases: neighbor generation, local search, and jump. Let N_k, $k = 1, \ldots, k_{max}$ be a set of pre-defined neighborhood structures and let $N_k(x)$ be the set of solutions in the kth-order neighborhood of a solution x. In the first phase, a neighbor $x' \in N_k(x)$ of the current solution is applied. Next, a solution x'' is obtained by applying local search to x'. Finally, the current solution jumps from x to x'' in case the latter improved the former. Otherwise, the order of the neighborhood is increased by one and the above steps are repeated until some stopping condition is satisfied.

A first attempt in this direction was done by Martins et al. [93]. The construction phase of their hybrid heuristic for the Steiner problem in graphs follows the greedy randomized strategy of GRASP, while the local search phase makes use of two different neighborhood structures as a VND strategy. Their heuristic was later improved by Ribeiro, Uchoa, and Werneck [121], one of the key components of the new algorithm being another strategy for the exploration of different neighborhoods. Ribeiro and Souza [120] also combined GRASP with VND in a hybrid heuristic for the degree-constrained minimum spanning tree problem. Canuto, Resende, and Ribeiro [33] used path-relinking in a GRASP for the prize collecting Steiner tree problem.

Festa et al. [54] studied different variants and combinations of GRASP and VNS for the MAX-CUT problem, finding and improving the best known solutions for some open instances from the literature.

Recent surveys on randomized metaheuristics can be found in [105].

4. Applications

Applications of GRASP can be grouped into the following two categories.

(1) Operations research problems. They include:

(a) routing [18, 22, 26, 35, 82];
(b) logic [42, 52, 102, 111, 113];
(c) covering and partition [16, 19, 46, 60, 71, 110];
(d) location [39, 40, 1, 41, 80, 132];
(e) minimum Steiner tree [34, 94, 92, 122];
(f) optimization in graphs [2, 21, 48, 83, 100, 115];
(g) assignment [4, 45, 58, 87, 95, 103, 104, 112, 127];
(h) timetabling and scheduling [5, 9, 15, 14, 28, 44, 49, 50, 84, 90, 123, 124, 125, 126]

(2) Industrial applications. They include:
(a) manufacturing [8, 24, 25, 31, 81, 98];
(b) transportation [18, 23, 45, 128];
(c) telecommunications [12, 13, 38, 80, 89, 106, 109, 129]
(d) graph and map drawing [53, 54, 86, 91, 115];
(e) power systems [29, 30, 77];
(f) computational biology [51, 68, 75];
(g) VLSI [17, 16], among other areas of application.

The reader can refer to [55, 56, 57], which contain annotated bibliographies of the GRASP literature from 1989 to 2008.

5. Concluding remarks

In this chapter, the basic components of GRASP methodology have been described. Improved and alternative solution construction mechanisms have been discussed as well as techniques proposed in literature for speeding up the search, hybridizations with other metaheuristics, and intensification and post-optimization strategies using path-relinking. It is difficult to formally analyze the quality of solution values found by using GRASP. However, there is an intuitive justification that views GRASP as a repetitive sampling technique, as explained in the following. Each GRASP iteration produces a sample solution from an unknown distribution of all obtainable results. The mean and variance of the distribution are functions of the restrictive nature of the candidate list. For example, if the RCL contains only one element, then only one solution will be produced and the variance of the distribution will be zero. Given an effective greedy function, the mean solution value in this case should be good, but probably suboptimal. If a less restrictive cardinality limit is imposed, many different solutions will be produced implying a larger variance. Since the greedy function is more compromised in this case, the mean solution value should degrade. Intuitively, however, by order statistics and the fact that the samples are randomly produced, the best value found should outperform the mean value. Indeed, often the best solutions sampled are optimal.

GRASP methodology has three further attractive characteristics: 1) it can be easily implemented; 2) contrary to other metaheuristics, in its basic version it needs the adjustment of the only the RCL parameter α; and 3) it can be trivially implemented in parallel. In the simplest parallel implementation of GRASP, each processor is initialized with its own copy of the procedure, the instance data, and an independent random number sequence.

Moreover, recent developments and hybridizations with other metaheuristics, presented in this chapter, show that different extensions to the basic GRASP allow further improvement to the solutions found. Among these, we highlight: Reactive

GRASP, which automates the adjustments of the restricted candidate list parameter; variable neighborhoods, which permit accelerated and intensified local search; and path-relinking, which allows the implementation of intensification strategies based on the memory of elite solutions.

The GRASP iterations can then be also performed in parallel with only a single global variable required to store the best solution found over all processors. GRASP has been applied to a wide range of combinatorial optimization problems, ranging from scheduling and routing to drawing and turbine balancing.

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