

A parallel multi-population genetic algorithm for a constrained two-dimensional orthogonal packing problem*

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This paper addresses a constrained two-dimensional (2D), non-guillotine restricted, packing problem, where a fixed set of small rectangles has to be placed into a larger stock rectangle so as to maximize the value of the rectangles packed. The algorithm we propose hybridizes a novel placement procedure with a genetic algorithm based on random keys. We propose also a new fitness function to drive the optimization. The approach is tested on a set of instances taken from the literature and compared with other approaches. The experimental results validate the quality of the solutions and the effectiveness of the proposed algorithm.

Keywords: Packing, cutting, two-dimensional packing, two-dimensional cutting, non-guillotine cutting, genetic algorithm.

1 Introduction

The constrained two-dimensional (2D), non-guillotine restricted, packing problem addressed in this paper consists of packing rectangular pieces into a large rectangular sheet

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of material (stock rectangle) in order to maximize the value of the rectangles packed. We consider the special case in which the items cannot be rotated and must be packed with their edges always parallel to the edges of the large rectangular sheet. The problem is relevant both from a theoretical point of view as well as practical one. It is NP-hard (Garey and Johnson, 1979) and arises in various production processes with applications varying from the home-textile to the glass, steel, wood, and paper industries, where rectangular figures are cut from large rectangular sheets of materials.

Various types of two-dimensional cutting problems have been considered in the literature. Dyckhoff (1990) provided a classification of the various types of cutting problems. Surveys for multidimensional cutting and packing problems are given by Dowsland and Dowsland (1992), Haessler and Sweeney (1991), and Sweeney and Paternoster (1992). Few authors have considered general two-dimensional non-guillotine cutting problems (constrained or unconstrained). The unconstrained non-guillotine cutting problem has been considered by a few authors, namely: Tsai et al. (1988) presented an integer programming approach; Arenales and Morabito (1995) developed an approach based on an AND/OR graph together with a branch and bound search; and Healy et al. (1999) introduced an algorithm based on the identification of the empty space that can be used to cut new items.

Optimal procedures for the constrained two-dimensional non-guillotine cutting problem have been proposed by several authors. Beasley (1985) proposed a branch and bound algorithm with an upper bound derived from a Lagrangian relaxation of a 0-1 integer linear programming formulation. Scheithauer and Terno (1993) presented an integer programming formulation in which binary variables are used to indicate whether a piece item is cut above/below or to the right/left of another piece. Hadjiconstantinou and Christofides (1995) developed a branch and bound algorithm in which the search was limited by using an upper bound based on a Lagrangian relaxation procedure and improved it using subgradient optimization. Computational gains were achieved by applying new reduction tests. Fekete and Schepers (1997a,b,c, 2004a,b) developed a two-level tree search algorithm for solving the d -dimensional knapsack problem. In this algorithm, projections of cut items were made onto both the horizontal and vertical edges of the stock rectangle. These projections were represented by graphs in which the nodes are the cut items and an edge joins two nodes if the projections of the corresponding cut items overlap. By looking at the properties of the graphs, the authors were able to check the feasibility of the corresponding patterns. Amaral and Letchford (2001) presented an upper bound which involved the solution of a large linear program by a column generation algorithm. Boschetti et al. (2002) proposed new upper bounds derived from different relaxations of a new integer programming formulation of the constrained two-dimensional non-guillotine cutting problem. The new formulation is based on the observation that any feasible solution can be represented by two sequences in which each element is the subset of items covering the x and y positions of the master surface, respectively. Caprara and Monaci (2004) compared four new algorithms based on the natural relaxation of the two-dimensional knapsack problem. In the relaxation, the knapsack has a capacity equal to the area of the master surface and the item weights are equal to their areas.

Heuristic procedures for the constrained two-dimensional non-guillotine cutting prob-

lem have also been developed by several authors. Lai and Chan (1997b) presented a heuristic based on simulated annealing. They used a problem representation which encodes the order in which pieces should be cut. Lai and Chan (1997a) proposed a heuristic based on an evolutionary strategy approach. They used the same representation as in Lai and Chan (1997b). Their algorithm includes an improvement procedure based on dividing the ordered list into active (cut) and inactive (uncut) pieces and seeing if any (currently) uncut pieces can be cut. Their mutation process involves swapping two pieces in the ordered list. Leung et al. (2001) used the ordered representation of Lai and Chan (1997b) and discussed producing a cutting pattern from it using both the difference process algorithm of Lai and Chan (1997b) and a standard bottom-left algorithm (see Jakobs (1996)). They showed that there exist problems for which the optimal solution cannot be found by these two approaches. They presented a simulated annealing heuristic with a move corresponding to either swapping two pieces in the ordered representation or moving a single piece to a new position in the representation. They also presented a genetic algorithm involving five different crossover operators. In an attempt to alleviate the problem of premature convergence, Leung et al. (2003) developed a hybrid heuristic blending simulated annealing with a genetic algorithm. Beasley (2004) proposes a genetic algorithm based on a nonlinear formulation of the problem, where variables indicate if a piece is cut or not and its position on the stock sheet. No placement algorithm is needed since the solutions are lists of variables and show directly the cutting pattern. Alvarez-Valdes et al. (2005) developed a new heuristic based on GRASP (Feo and Resende, 1989, 1995) for the non-guillotine two-dimensional cutting stock problem. The constructive phase explicitly considers the possibility of simultaneously cutting several pieces of the same type and forming a block, as one would do in the pallet loading problem. Alvarez-Valdes et al. (2007) propose a tabu search algorithm that define several moves based on reducing and inserting blocks of pieces and include intensification and diversification procedures based on long-term memory.

In this paper, we present a hybrid heuristic for a two-dimensional non-guillotine cutting problem which combines a random-keys based genetic algorithm with a novel fitness function and a new heuristic placement policy. The remainder of the paper is organized as follows. In Section 2, we define the problem formally, and in Section 3, propose the new approach, describing the genetic algorithm, the placement strategy, and the fitness function. In Section 4, we present experimental results and in Section 5 make concluding remarks.

2 The problem

The two-dimensional packing problem addressed in this paper is the problem of packing into a single large planar stock rectangle (W, H) , of width W and height H , smaller rectangles (w_i, h_i) , $i = 1, \dots, n$, each of width w_i and height h_i . Each rectangle i has a fixed orientation (i.e. cannot be rotated); must be packed with its edges parallel to the edges of the stock rectangle; and the number x_i of pieces of each rectangle type that are to be packed must lie between P_i and Q_i , i.e. $0 \leq P_i \leq x_i \leq Q_i$, for all $i = 1, \dots, n$.

Each rectangle $i = 1, \dots, n$ has an associated value equal to v_i and the objective is to maximize the total value of the rectangles cut $\sum_{i=1}^n v_i x_i$. Without significant loss of generality, it is usual to assume that all dimensions W, H , and (w_i, h_i) , $i = 1, \dots, n$, are integers. To simplify notation, we shall use $M = \sum_{i=1}^n Q_i$ as the maximum number of rectangles that can be packed. According to the newly developed typology by Wäscher et al. (2007), the problem falls into the *output maximization assignment* kind and can be classified as a 2D-SKP problem type (two-dimensional single knapsack problem) given that it handles instances where a strong heterogeneous assortment of small items exist.

Depending on the values of P_i and Q_i , we can distinguish the following three types of problems:

1. In the *unconstrained* type, we have $P_i = 0$, $Q_i = \lfloor (W \times H)/(w_i \times h_i) \rfloor$, for $i = 1, \dots, n$, which is a trivial bound;
2. In the *constrained* type, we have: for all $i = 1, \dots, n$, $P_i = 0$ and $\exists j \in \{1, \dots, n\}$ such that $Q_j < \lfloor (W \times H)/(w_j \times h_j) \rfloor$;
3. In the *doubly constrained* type, we have: $\exists i \in \{1, \dots, n\}$ such that $P_i > 0$ and $\exists j \in \{1, \dots, n\}$ such that $Q_j < \lfloor (W \times H)/(w_j \times h_j) \rfloor$.

A simple upper bound for the problem can be obtained by solving the following bounded knapsack problem, where variable x_i represents the number of pieces of type i to be cut in excess of its lower bound P_i :

$$\begin{aligned} & \max \sum_{i=1}^n v_i x_i + \sum_{i=1}^n v_i P_i \\ & \text{subject to} \\ & \sum_{i=1}^n (w_i \cdot h_i) x_i \leq W \cdot H - \sum_{i=1}^n (w_i \cdot h_i) P_i \\ & x_i \leq Q_i - P_i, \quad i = 1, \dots, n, \\ & x_i \geq 0, \text{ integer}, \quad i = 1, \dots, n. \end{aligned}$$

This simple upper bound will be used to evaluate the performance of the hybrid genetic algorithm described in this paper.

3 New approach

3.1 Overview about the new approach

The new approach proposed in this paper combines a random-keys based multi population genetic algorithm, a new placement strategy, and a novel measure of solution quality that we call *modified total value*. Instead of describing the algorithms directly as cutting problems, we use the equivalent notion of packing, where we are given rectangles and wish to pack them on the stock rectangle.

The role of the genetic algorithm is to evolve the encoded solutions, or *chromosomes*, which represent the rectangle packing sequence and the type of placement procedure used to place each rectangle. For each chromosome, the following four phases are applied:

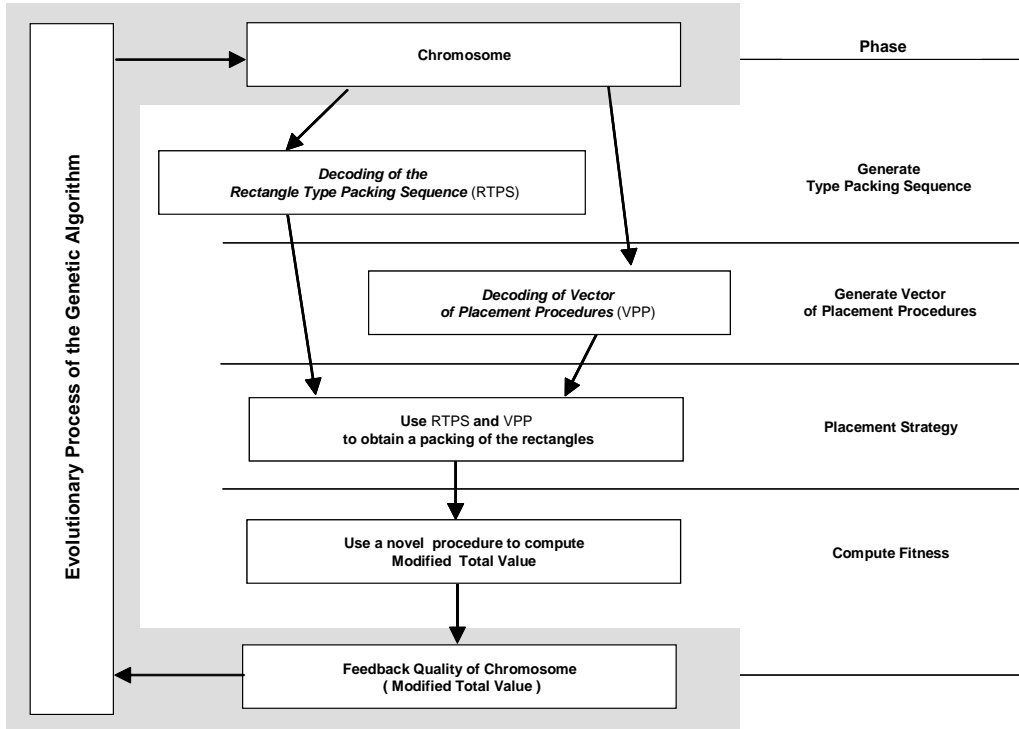


Figure 1: Architecture of the new approach.

1. *Decoding of the rectangle packing sequence.* The first phase is responsible for transforming part of the chromosome supplied by the genetic algorithm into the sequence in which the rectangles are to be packed in the stock rectangle.
2. *Decoding of the placement procedure.* In the second phase, part of the chromosome supplied by the genetic algorithm is transformed into a vector that indicates which placement procedure is to be used to position each rectangle in the stock rectangle.
3. *Placement strategy.* The third phase makes use of the rectangle packing sequence defined in the first phase and the vector of placement procedures defined in second phase and constructs a packing of the rectangles.
4. *Fitness evaluation.* In the final phase, we make use of a novel procedure to compute a modified total value which is used as a fitness measure (quality measure) to feedback to the genetic algorithm.

Figure 1 illustrates the sequence of steps applied to each chromosome generated by the genetic algorithm.

The remainder of this section describes in detail the genetic algorithm, the placement strategy, and the fitness function.

3.2 Genetic algorithm

The following sub sections present the chromosome representation and decoding, the evolutionary process, the initialization process and the multi-population strategy.

3.2.1 Chromosome representation and decoding

The genetic algorithm described in this paper uses a random-keys alphabet comprised of random real numbers between 0 and 1. The evolutionary strategy used is similar to the one proposed by Bean (1994), the main difference occurring in the crossover operator. The important feature of random keys is that all offspring formed by crossover are feasible solutions. This is accomplished by moving much of the feasibility issue into the objective function evaluation. If any random-key vector can be interpreted as a feasible solution, then any crossover vector is also feasible. Through the dynamics of the genetic algorithm, the system learns the relationship between random-key vectors and solutions with good objective function values.

A chromosome represents a solution to the problem and is encoded as a vector of random keys. In a direct representation, a chromosome represents a solution of the original problem, and is called *genotype*, while in an indirect representation it does not, and special procedures are needed to derive from it a solution called a *phenotype*. In the present context, the direct use of cutting patterns as chromosomes is too complicated to represent and manipulate. In particular, it is difficult to develop corresponding crossover and mutation operations. Instead, solutions are represented indirectly by parameters that are later used by a decoding procedure to obtain a solution. To obtain the solution (phenotype) we use the placement strategy to be described in Section 3.3.

Recall that there are n rectangle types and that at most Q_i rectangles of type i can be packed into the stock rectangle. In the description of the genetic algorithm, we take a total of $M = \sum_{i=1}^n Q_i$ rectangles, i.e. Q_i rectangles of type $i = 1, \dots, n$.

Each solution chromosome is made of $2M$ genes, where M is the number of rectangles to be packed, i.e.

$$Chromosome = (\underbrace{gene_1, \dots, gene_M}_{\text{Rectangle Type Packing Sequence}}, \underbrace{gene_{M+1}, \dots, gene_{2M}}_{\text{Vector of Placement Procedures}}).$$

The first M genes are used to obtain the *Rectangle Type Packing Sequence* (RTPS) while the last M genes are used to obtain the *Vector of Placement Procedures* (VPP).

Both the RTPS and VPP are used by the placement strategy. The decoding (mapping) of the first M genes of each chromosome into an RTPS is accomplished by sorting the genes and rectangles types in ascending order. Figure 2 presents an example of the decoding process for the RTPS. In the example there are four types of rectangles and the values of Q_i are $Q_1 = 2, Q_2 = 3, Q_3 = 1$ and $Q_4 = 2$. According to the ordering obtained the rectangles types should be packed in the following order 2, 4, 2, 1, 2, 1, 3, 4.

In the placement strategy we make use of two placement procedures; *BL* (bottom-left) and *LB* (Left-Bottom) (see Section 3.3). The decoding (mapping) of the last M genes

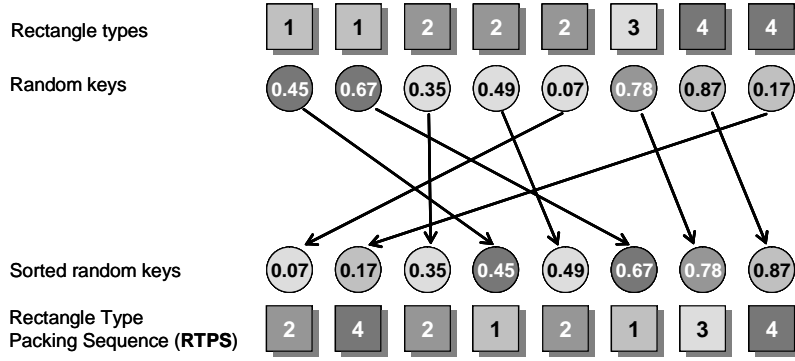


Figure 2: Chromosome decoding procedure.

of each chromosome into a VPP is accomplished using, for $i = 1, \dots, M$, the following expression:

$$VPP(i) = \begin{cases} BL & \text{if } gene(M + i) \leq \frac{1}{2}, \\ LB & \text{if } \frac{1}{2} < gene(M + i). \end{cases}$$

Note that with this representation we are able to handle the upper bounds $x_i \leq Q_i$, $i = 1, \dots, n$, implicitly. Later in Section 3.4 we discuss how we deal with the lower bounds $x_i \geq P_i$, for one or more $i \in \{1, \dots, n\}$, in the case of doubly constrained problems.

3.2.2 Evolutionary process

To breed good solutions, the random-key vector population is operated upon by a genetic algorithm. There are many variations of genetic algorithms obtained by altering the reproduction, crossover, and mutation operators. The reproduction and crossover operators determine which parents will have offspring, and how genetic material is exchanged between the parents to create those offspring. Mutation allows for random alteration of genetic material. Reproduction and crossover operators tend to increase the quality of the populations and force convergence. Mutation opposes convergence and replaces genetic material lost during reproduction and crossover.

The *population is initialized* with random-key vectors whose components are random real numbers uniformly sampled from the interval $[0, 1]$. *Reproduction* is accomplished by first copying some of the best individuals from one generation to the next, in what is called an *elitist strategy* (Goldberg, 1989). The advantage of an elitist strategy over traditional probabilistic reproduction is that the best solution is monotonically improving from one generation to the next. The potential downside is population convergence to a local minimum. This can, however, be overcome by an appropriate amount of mutation as described below.

Parameterized uniform crossovers (Spears and Dejong, 1991) are employed in place of the traditional one-point or two-point crossover. After two parents are chosen, one chosen randomly from the best (unlike Bean (1994) we always choose one parent from the

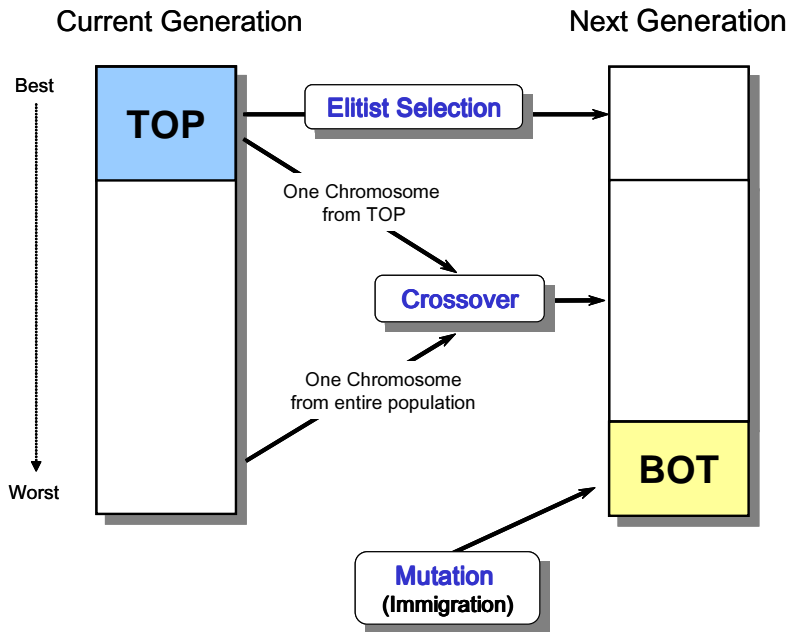


Figure 3: Transitional process between consecutive generations.

best, Gonçalves and Resende (2009) show that this change produces results with better quality and faster) and the other chosen randomly from the full old population (including chromosomes copied to the next generation in the elitist selection), at each gene we toss a biased coin to select which parent will contribute the allele, *see TOP* in Figure 3. Figure 4 presents an example of the crossover operator. It assumes that a coin toss of heads selects the gene from the first parent, a tails chooses the gene from the second parent, and that the probability of tossing a heads, crossover probability $CProb = 0.7$. In Section 4 we describe how we determine this value empirically.

Rather than using the traditional gene-by-gene mutation with very small probability at each generation, some new members of the population are randomly generated from the same distribution as the initial population (*see BOT* in Figure 3). The purpose of this process is to prevent premature convergence of the population, like in a mutation operator, and leads to a simple statement of convergence. Figure 3 depicts the transitional process between two consecutive generations.

3.2.3 Population Initialization

In our approach the initial population is not fully randomly generated. We introduce in the initial population four non-random chromosomes. These non-random chromosomes are created so that their corresponding RTPSs are equivalent to packing the rectangles in descending order of their value, i.e., rectangles with higher value appear first in the packing sequence. The four chromosomes result from considering four variations for the placement procedures: random, all BL, all LB and alternatively BL and LB. As

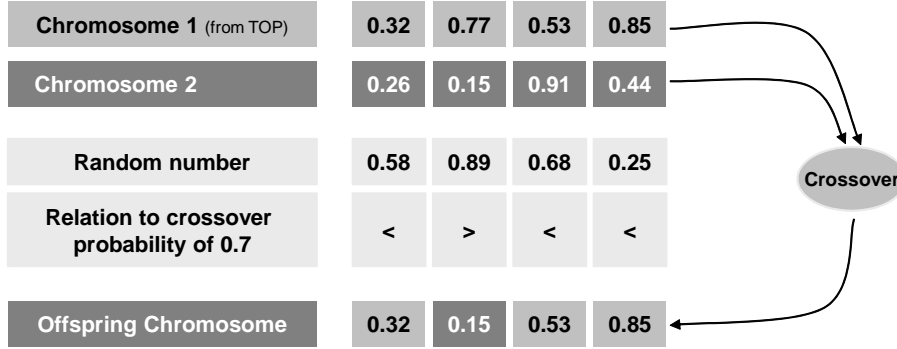


Figure 4: Example of parameterized uniform crossover with crossover probability equal to 0.7.

will be shown in the numerical experiments the introduction of the four non-random chromosomes improves significantly the quality of the solutions obtained.

3.2.4 Multi-population strategy

In our multi-population strategy several populations are evolved independently. After a pre-determined number of generations all the populations exchange information. The information exchanged is the chromosomes of good quality. When evaluating possible interchange strategies we noticed that exchanging too much information (exchanging too many chromosomes) leads to the disruption of the evolutionary process. Also, if the populations exchange information with a high frequency they do not have enough time to produce good results because their evolutionary process is disrupted before good solutions can be achieved. Having this information in mind we chose a multi-population strategy that after a pre-determined number of generations (this will be determined empirically in Section 4) selects the two best chromosomes from all current chromosomes and then copies them to the populations where they don't currently exist and removes the corresponding number of the worst chromosomes.

3.3 Placement strategy

Rectangles types are placed in the stock rectangle, one at a time, in the order defined by the RTPS supplied by the genetic algorithm (see Section 3.2.2). While trying to place a rectangle type in the stock rectangle we consider only maximal *empty rectangular spaces* (ERSs), i.e. ERSs that are not contained in any other ERS. A rectangle type is placed in an ERS where it fits and according to the placement procedure defined for its placement by the genetic algorithm.

Initially we considered only a *Bottom-Left* (BL) placement procedure which first orders the ERSs in such way that $ERS_i < ERS_j$ if $y_i < y_j$ or $y_i = y_j, x_i < x_j$ and then chooses the first ERS in which the rectangle type to be packed fits (note that x_i/y_i is the coordinate x/y of the bottom left corner of the ERS), see pseudo-code in Figure 5.

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procedure BL( $r_i$ )
1  Let  $r_i$  be a rectangle of type  $i$  to be packed in the stock rectangle;
2  Let  $N_{ERS}$  be the number of available ERSs;
3  Initialize  $L \leftarrow W$ ;  $B \leftarrow H$ ;
4  for  $k = 1, \dots, N_{ERS}$  do
5      Let  $x(ERS_k)$  be the  $x$  coordinate of the the bottom left corner of  $ERS_k$ ;
6      Let  $y(ERS_k)$  be the  $y$  coordinate of the the bottom left corner of  $ERS_k$ ;
7      if  $r_i$  fits in  $ERS_k$  then
8          if  $x(ERS_k) \leq L$  or ( $x(ERS_k) = L$  and  $y(ERS_k) \leq B$ ) then
9               $L \leftarrow x(ERS_k)$ ;  $B \leftarrow y(ERS_k)$ ;
10              $ERS_i^* = ERS_k$ 
11         end if
12     end if
13 end for
14 Return  $ERS_i^*$ ;
end BL;

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Figure 5: Pseudo-code of the Bottom-Left (BL) placement procedure.

However, as observed by Liu and Teng (1999), we noticed that some optimal solutions could not be constructed by the BL placement procedure. In other words, given an optimal solution to a problem, it is possible that no RTPS exists that, combined with the BL placement procedure, produces the given optimal solution. Figure 6 shows an optimal solution for one problem where the BL placement procedure cannot find the optimal solution.

In Figure 7, we present, for the problem in Figure 6, all the solutions obtained by the BL placement procedure for all the possible RTPSs starting with rectangle 2 (Similar sub-optimal solutions are produced when rectangles 1, 3, and 4 are fixed in the first position). As can be observed, none of those RTPSs, when combined with BL placement procedure, produces the optimal solution in Figure 6.

To overcome this weakness, we combine the BL placement procedure with a *Left-Bottom* (LB) placement procedure which first orders the ERSs in such way that $ERS_i < ERS_j$ if $x_i < x_j$ or $x_i = x_j, y_i < y_j$ and then chooses the first ERS in which the rectangle type to be packed fits (note that x_i/y_i is the coordinate x/y of the bottom left corner of the ERS). In summary, our placement strategy uses two placement procedures, the Bottom-Left and the Left-Bottom, to construct a packing of the rectangle types. The vector of placement procedures (VPP), supplied by the genetic algorithm, indicates, for each rectangle type to be packed, whether it should be placed using the BL or LB procedure. In Figure 6 we present the optimal solution found using $RTPS = (2, 1, 4, 3)$ and $VPP = (BL, BL, LB, BL)$.

When placing a rectangle type in a ERS we try to build a layer containing several rectangles of that rectangle type. We use two types of layers: horizontal and vertical. When we use the BL placement procedure we try to create a horizontal layer and when

$$\frac{RTPS = 2 - 1 - 4 - 3}{VPP = BL - BL - LB - BL}$$

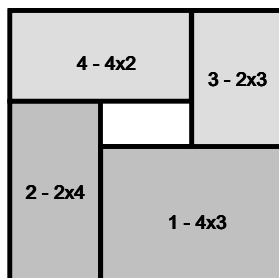


Figure 6: Optimal solution found by combining the Bottom-Left (BL) and the Left-Bottom (LB) placement procedures.

we use the LB placement procedure we try to create a vertical layer (see Figure 8).

Let i be the type of m^{th} , $m = 1, \dots, M$, rectangle to be placed as defined by the RTPS. Furthermore, let VPP_m be the placement procedure that the genetic algorithm assigned for the placement of m^{th} rectangle type. The pseudo-code of the placement procedure is presented in Figure 9. To generate and keep track of the ERSs, we make use of the *Difference Process* (DP), developed by Lai and Chan (1997b).

3.4 Modified total value fitness function

A natural fitness function (measure of quality) for this type of problem is the *total value* given by:

$$Total\ value = \sum_{i=1}^n v_i x_i,$$

where x_i is the number of rectangular pieces of type i to be cut in a solution and v_i is the value of each rectangular piece of type i . This measure, however, is not ideal because it does not capture well the potential for improvement of a solution. To be able to capture the improvement potential of different packings which have the same total value, we use a fitness measure that we call *modified total value* which is an adaptation of the fitness measure proposed by Gonçalves (2007) and is given by the following expression:

$$Modified\ Total\ Value = Total\ Value + 0.03 \times \frac{Minimum\ Value\ Rectangle \times Area\ of\ Largest\ ERS}{Area\ of\ Stock\ Rectangle}.$$

In the form presented above, our algorithm does not handle doubly constrained problems since lower bounds are not enforced. To enable the algorithm to handle these problems, we made one further modification to the fitness function. The lower bounds

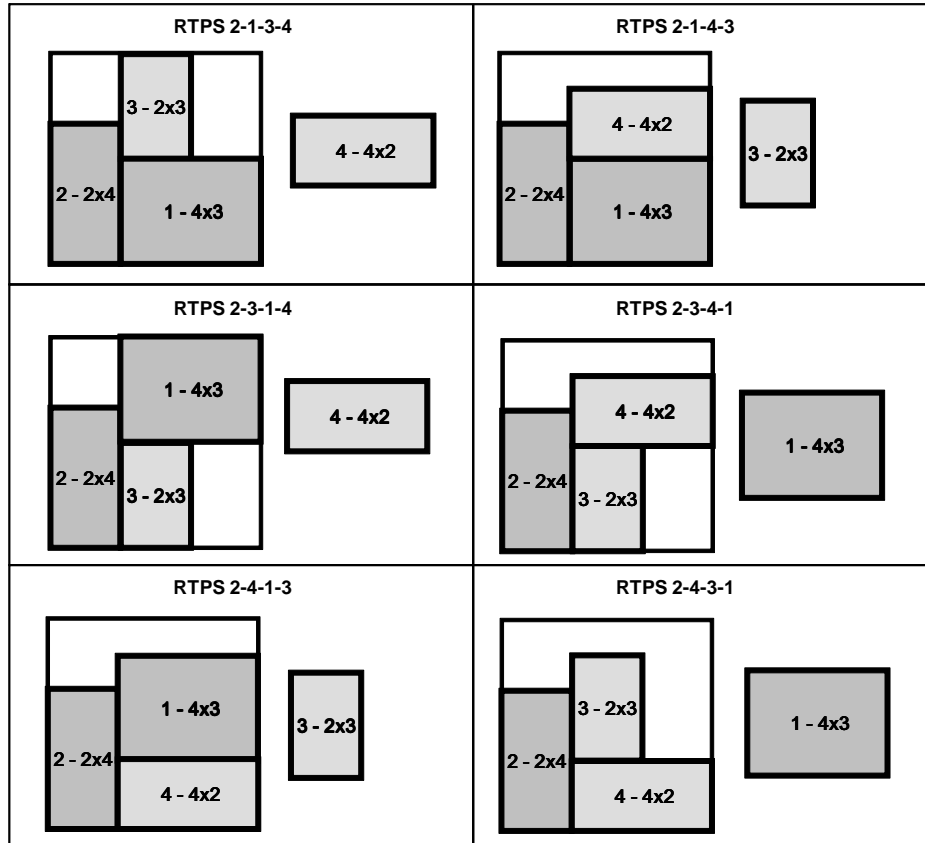
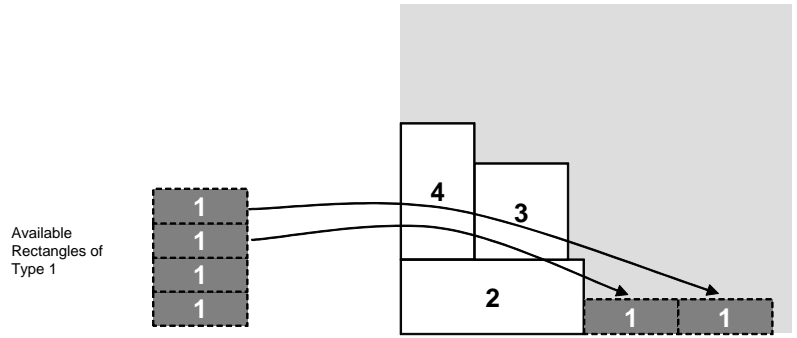


Figure 7: Solutions found by the Bottom-Left (BL) placement procedure for the problem in Figure 6 for all Rectangle Packing Sequences with rectangle 2 in the first position.

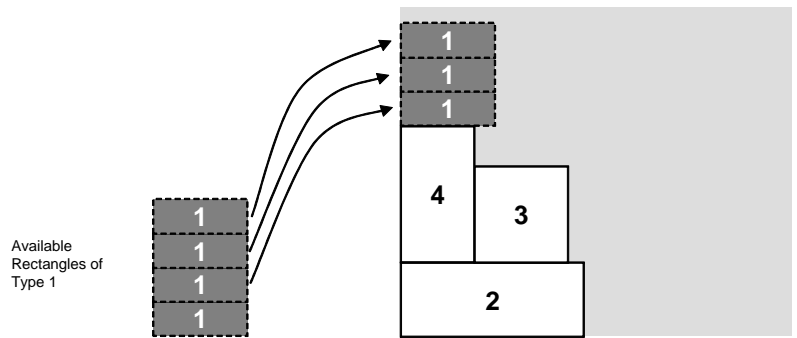
P_i for each rectangle of type i are treated indirectly through the use of a penalty term in the fitness function. If any lower bound P_i is not satisfied in a solution, then a penalty parameter PF is subtracted from the modified total value. All the computational tests were performed using $PF = 10^{10}$.

3.5 Parallel implementation

In our parallel implementation we parallelized only the task that performs the evaluation of the chromosomes fitness since it is the most time consuming. The tasks related with the GA logic were not parallelized since they consume very little time. This type of parallelization is easy to implement and in multi-core CPUs allows for a large reduction in computational times (almost a linear speed-up with the number of cores). The parallel version of our approach was implemented using the OpenMP Application Program Interface (API) which supports multi-platform shared-memory parallel programming in C/C++.



a) Packing a horizontal layer of type 1 rectangles using the **Bottom-Left** procedure



b) Packing a vertical layer of type 1 rectangles using the **Left-Bottom** procedure

Figure 8: Packing of a layer using a) the Bottom-Left procedure and b) the Left-Bottom procedure.

4 Numerical experiments

In this section we report the results obtained on a set of experiments conducted to evaluate the performance of the Multi-Population Genetic Algorithm (MPGA) proposed in this paper.

4.1 Benchmark algorithms

We compare the MPGA with the following four recently proposed heuristics, which present the best computational results to date:

- *PH* – A population heuristic, proposed by Beasley (2004), where a population of solutions to the problem is progressively evolved.

Table 1: Range of Parameters in past implementations.

Parameter	Interval
TOP	0.10 - 0.25
BOT	0.15 - 0.30
Crossover Probability (CProb)	0.70 - 0.80

- *GA* – Proposed by Hadjiconstantinou and Iori (2007), this genetic algorithm uses an elitist theory, immigration, online heuristics, and tailored crossover operators.
- *GRASP* – A greedy randomized adaptive search procedure proposed by Alvarez-Valdes et al. (2005).
- *TABU* – A tabu search approach proposed by Alvarez-Valdes et al. (2007).

4.2 Test problem instances

The effectiveness of MPGA is evaluated by solving the same four sets of test problem instances used by Alvarez-Valdes et al. (2007). These problem instances sets are:

1. A set of 21 problem instances taken from the literature: 15 from Beasley (1985), two from Hadjiconstantinou and Christofides (1995), one from Wang (1983), one from Christofides and Whitlock (1977), and five from Fekete and Schepers (2004c). All these problem instances have known optimal solutions.
2. A set of 630 large random problems generated by Beasley (2004), following Fekete and Schepers (2004c).
3. A set of 31 zero-waste problem instances used by Leung et al. (2003). I
4. A set of 21 doubly constrained problems. These problem instances are the result of transformation of the first set by Beasley (2004) into doubly constrained problems by defining, for some types of rectangles, non-zero lower bounds.

4.3 GA configuration

Configuring genetic algorithms is oftentimes more an art form than a science. In our past experience with genetic algorithms based on the same evolutionary strategy (see Gonçalves and Almeida (2002), Ericsson et al. (2002), Gonçalves and Resende (2004), Gonçalves et al. (2005), Buriol et al. (2005), Buriol et al. (2007), and Gonçalves (2007)), we obtained good results with values of TOP, BOT, and Crossover Probability (CProb) in the intervals shown in Table 1.

For the population size, we have obtained good results by indexing it to the size of the problem, i.e. use small size populations for small problems and larger populations for larger problems. With this in mind and to obtain a reasonable configuration we conducted a small pilot study including 21×3 instances from the set with large random

Table 2: Configuration used on all runs in computational experiments.

Population size	$\min \{15 \times \text{number of input rectangles}, 2000\}$
Crossover probability	0.7
TOP	The 25 % most fit chromosomes from the previous generation are copied to the next generation
BOT	The 15 % least fit chromosomes from the previous generation are replaced with randomly generated chromosomes
Number of populations	3
Exchange of information between populations	Every 15 generations
Fitness	Maximize modified total value using $k=0.03$
Stopping Criterion	Stop after 1000 generations

problems and 4 instances from the set with zero-waste problems. We tested all the combinations of the following values:

- $TOP \in \{0.10, 0.15, 0.20, 0.25\}$;
- $BOT \in \{0.15, 0.20, 0.25, 0.30\}$;
- $CProb \in \{0.70, 0.75, 0.80\}$;
- *Population size* with 2, 5, 10 and 15 times the number of rectangles in the problem instance.

For each of the 192 possible configurations, we made three independent runs of the algorithm (with three distinct random number generator seeds) and computed the average total value. The configuration that minimized the sum, over the pilot problem instances, was $TOP = 25\%$, $BOT = 15\%$, $CProb = 0.7$, and *Population size* = $15 \times$ the number of rectangles in the problem instance.

To determine the appropriate value of K for the modified total value, we tested the algorithm using values of K between 0.01 and 0.10, in steps of 0.01, on the instances of the pilot problem instances. We made three independent runs of the algorithm, using the best configuration determined previously, and computed the average modified total value. The value of K that maximized the sum, over all the pilot problem instances, of the average modified total packing value was chosen, i.e. $K = 0.03$.

After some experimentation with the problem instances in a pilot study we come to the conclusion that using 3 parallel populations and exchanging information every 15 generations was the best configuration for this type of problem.

The configuration presented in Table 2 was held constant for all experiments and all problems instances.

The computational results presented in the next section demonstrate that this configuration not only provides excellent results in terms of solution quality but also is very robust.

Table 3: Versions of the approach.

Version	Description
BL	using only the Bottom Left procedure
BL-L	using only the Bottom Left procedure with Layers
BL-LB-L	using only the Bottom Left and Left Bottom procedures with Layers
BL-LB-L-4NR	using the Bottom Left and Left Bottom procedures with Layers and with the introduction of four non-random chromosomes in the first population

Table 4: Overall average percentage deviation from optimum / lower bound per version.

Set n ^o	Description	BL	BL-L	BL-LB-L	BL-LB-L-4NR
1	Problems from literature	0	0	0	0
2	Large random problems	1.04	1.00	0.87	0.83
3	Zero-waste problems	0.48	0.48	0.24	0.17
4	Doubly constrained problems	6.36	6.36	6.36	6.36

4.4 Computational results

Our algorithm (MPGA) was implemented in C++ and the computational experiments were carried out on a computer with a Intel 2.66GHz Xeon Quadcore CPU with the Linux CentOS 5 operating system.

Before presenting the final and detailed computational results we will present a summary of the the impact of each of the different changes that we introduced in our approach. We compare the impact of each change using the four versions presented in table 3.

Table 4 presents for each version the overall average percentage deviation from optimum / lower bound. Test set n^o 1 and n^o 4 consist of very samll instances and so even the most basic version **BL** is able to obtain the best values. The impact of using layers is positive and can only be seen for test set n^o 3 because it is the only set having more than one rectangle per type. Finally, the results for test sets n^o 2 and n^o 3 show clearly that the combination of BL with LB and the introduction of four non-random chromosomes in the initial population provide a very positive impact on the overall average percentage deviation from optimum / lower bound.

The complete computational results for version BL-LB-L-4NR appear in Tables 5-8. All tests where performed using the configuration summarized in Table 2. In terms of computational times we cannot make any fair and meaningful comments since all the other approaches were implemented with different programming languages and tested on computers with different computing power. Instead, we limit ourselves to reporting the average running times for MPGA.

Table 5 includes a direct comparison with the results for PH (Beasley, 2004), GA

(Hadjiconstantinou and Iori, 2007), GRASP (Alvarez-Valdes et al., 2005), and TABU - Alvarez-Valdes et al. (2007) in terms of solution quality. The results show that the MPGA algorithm, finds the optimal solutions for all the problem instances and is, therefore, as good as algorithms GA and TABU.

The results for the large random problems are presented in Table 6. This table displays aggregate results, showing that MPGA produces overall average deviations from the upper bound for all problem types (Type I, Type II, and Type III) that are always lower than those produced by all the other heuristics (see bottom of table 6). A close look at the results shows that MPGA outperforms heuristics PH and GA for all problem types and sizes. From the table it is also clear that MPGA outperforms the GRASP and TABU heuristics not only because it obtained better average deviations from the upper bound than both (GRASP=1.07%, TABU=0.98% and MPGA=**0.83%**) but also because it obtained better number of best results for the 21 combinations of sizes and types (GRASP=5/21, TABU=8/21 and MPGA=**20/21**).

Table 7 presents a direct comparison between MPGA and the GRASP of Alvarez-Valdes et al. (2005) and the TABU of Alvarez-Valdes et al. (2007) on the set of zero-waste problem instances (the other benchmark heuristics have not run these instances). MPGA produces overall average deviations from the optimal values that are lower than those produced by the GRASP and TABU heuristics (GRASP=1.68%, TABU=0.42% and MPGA=**0.17%**). In terms of the number of best results over all the instance in this set MPGA clearly outperforms the other two (GRASP=5/31, TABU=17/31, MPGA=**30/31**).

Finally, Tables 8 show the results for the doubly-constrained test problem instance of the fourth set, where the heuristics PH (Beasley, 2004), GRASP (Alvarez-Valdes et al., 2005), TABU Alvarez-Valdes et al. (2007) and MPGA are contrasted. The upper bound (UB) corresponds to the solution of the constrained problem using the knapsack problem formulation of Section 2. The problems for which the algorithms do not find solutions are not feasible, but are maintained in the set of test problems and therefore are included in the table. For these problem instances, MPGA clearly outperforms the other two both in terms of average deviation from the lower bound (PH=8.11%, GRASP=7.36%, TABU=6.62%, MPGA=**6.36%**) as well as in terms of the number of best solutions obtained out of the 21 problems (11/21, 12/21, 17/21, **19/21**).

```

procedure PLACEMENT
1  Let  $R_i$  be the remaining quantity of unpacked rectangles of type  $i$  ;
2  Initialize  $R_i \leftarrow Q_i$   $i = 1, \dots, n$ ;
3  for  $m = 1, \dots, M$  do
4      Let  $i = RTPS(m)$  , be the type of  $m^{th}$  rectangle to be packed
      .
      according to the RTPS;
5      if  $R_i > 0$  then // check if there any rectangles of type  $i$  remaining to pack;
6           $ERS_i^* \leftarrow \emptyset$ 
7           $nL \leftarrow \emptyset$ 
8          if  $VPP_m = BL$  then
9              Let  $ERS_i^*$  be the ERS, in the list of available ERSs,
          .
          in which a rectangle of type  $i$  is placed when the Bottom-Left
          .
          placement heuristic is applied;
10         else if  $VPP_m = LB$  then
11             Let  $ERS_i^*$  be the ERS, in the list of available ERSs,
          .
          in which a rectangle of type  $i$  is placed when the Left-Bottom
          .
          placement heuristic is applied;
12         end if
13         if  $ERS_i^* > \emptyset$  then // an ERS was found ;
14             if  $VPP_m = BL$  then
15                 Let  $nL =$  number of rectangles of type  $i$ 
          .
          that can be placed in a horizontal layer in  $ERS_i^*$ 
          .
          (see Figure 8a);
16                 Place a horizontal layer with  $nL$  rectangles of type  $i$ 
          .
          at the bottom left corner of  $ERS_i^*$ ;
17             else if  $VPP_m = LB$  then
18                 Let  $nL =$  number of rectangles of type  $i$ 
          .
          that can be placed in a vertical layer in  $ERS_i^*$ 
          .
          (see Figure 8b);
19                 Place a vertical layer with  $nL$  rectangles of type  $i$ 
          .
          at the bottom left corner of  $ERS_i^*$ ;
20             end if
21             Update the list of available ERSs using the DP process
          .
          of Lai and Chan (1997b);
22         end if
23          $R_i \leftarrow R_i - nL$ 
24     end if
25 end for
end PLACEMENT;

```

Figure 9: Pseudo-code of the placement procedure of the hybrid heuristic.

Table 5: Computational results - problems from literature.

Source of problem	Inst.	Dim. (L×W)	m	M	Opt.	PH	GA	GRASP	TABU	MPGA	CPU Time (s)
Beasley (1985)	1	10x10	5	10	164	164	164	164	164	164	0.00
	2	10x10	7	17	230	230	230	230	230	230	0.00
	3	10x10	10	21	247	247	247	247	247	247	0.00
	4	15x10	5	7	268	268	268	268	268	268	0.00
	5	15x10	7	14	358	358	358	358	358	358	0.01
	6	15x10	10	15	289	289	289	289	289	289	0.03
	7	20x20	5	8	430	430	430	430	430	430	0.01
	8	20x20	7	13	834	834	834	834	834	834	0.01
	9	20x20	10	18	924	924	924	924	924	924	0.01
	10	30x30	5	13	1452	1452	1452	1452	1452	1452	0.01
	11	30x30	7	15	1688	1688	1688	1688	1688	1688	0.01
	12	30x30	1	22	1865	1801	1865	1865	1865	1865	0.04
Hadjiconstantinou and Christofides (1995)	3	30x30	7	7	1178	1178	1178	1178	1178	1178	0.01
	11	30x30	15	15	1270	1270	1270	1270	1270	1270	0.01
Wang (1983)		70x40	19	42	2726	2721	2726	2726	2726	2726	0.02
Christofides and Whitlock (1977)	3	40x70	20	62	1860	1720	1860	1860	1860	1860	0.05
Fekete and Schepers (1997a)	1	100x100	15	50	27718	27486	27718	27589	27718	27718	0.05
	2	100x100	30	30	22502	21976	22502	21976	22502	22502	0.14
	3	100x100	30	30	24019	2343	24019	23743	24019	24019	0.55
	4	100x100	33	61	32893	31269	32893	32893	32893	32893	0.06
	5	100x100	29	97	27923	26332	27923	27923	27923	27923	0.02
Mean Percentage deviation from upper bound						5.49	0	0.19	0	0	

Table 6: Computational results - Large random problems.

m	Qt per type	M	PH	GA	GRASP	TABU	MPGA	CPU Time (s)
40	1	40	7.77	6.12	6.97	6.55	5.96	1.78
	3	120	3.54	2.82	2.22	1.95	1.60	6.39
	4	160	3.24	2.40	1.81	1.65	1.40	12.94
50	1	50	5.48	4.56	4.80	4.85	4.25	2.69
	3	150	2.35	1.89	1.50	1.27	1.03	10.90
	4	200	2.63	1.86	1.18	0.96	0.71	22.90
100	1	100	2.26	1.69	1.51	1.50	1.17	11.81
	3	300	1.27	0.99	0.47	0.31	0.21	40.72
	4	400	1.06	0.85	0.26	0.18	0.11	22.70
150	1	150	1.31	1.06	0.89	0.84	0.55	23.44
	3	450	0.60	0.32	0.14	0.07	0.05	37.56
	4	600	0.92	0.60	0.11	0.05	0.03	47.07
250	1	250	0.88	0.75	0.51	0.45	0.25	39.11
	3	750	0.57	0.51	0.04	0.01	0.01	72.70
	4	1000	0.39	0.28	0.03	0.00	0.00	20.40
500	1	500	0.26	0.21	0.07	0.00	0.03	54.81
	3	1500	0.18	0.19	0.00	0.00	0.00	6.07
	4	2000	0.18	0.19	0.00	0.00	0.00	4.71
1000	1	1000	0.09	0.15	0.00	0.00	0.00	47.24
	3	3000	0.07	0.12	0.00	0.00	0.00	8.26
	4	4000	0.07	0.17	0.00	0.00	0.00	6.75
Type I			1.64	1.24	1.04	0.95	0.86	21.27
Type II			1.70	1.37	1.14	1.06	0.88	23.50
Type III			1.66	1.35	1.03	0.94	0.74	26.80
All			1.67	1.32	1.07	0.98	0.83	23.86
Mean percentage deviations from knapsack upper bound.								

Table 7: Computational results - zero-waste problems.

Source of problem	Inst.	Dim. (L x W)	m	M	Opt.	GRASP	TABU	MPGA	CPU Time (s)
Lai and Chan (1997b)	1	400x200	9	10	80000	80000	80000	80000	0.67
	2	400x200	7	15	79000	79000	79000	79000	0.01
	3	400x200	5	20	160000	154600	160000	160000	3.10
Jakobs (1996)	1	70x80	14	20	5600	5447	5600	5600	0.58
	2	70x80	16	25	5600	5455	5540	5540	0.70
	3	120x45	22	25	5400	5328	5400	5400	11.52
	4	90x45	16	30	4050	3978	4050	4050	3.47
	5	65x45	18	30	2925	2871	2925	2925	5.19
Leung et al. (2003)	1	150x110	40	40	16500	15856	16280	16340	23.71
	2	160x120	50	50	19200	18628	19044	19116	15.18
Hopper and Turton (2001)	1-1	20x20	16	16	400	400	400	400	0.29
	1-2	20x20	17	17	400	386	400	400	0.82
	1-3	20x20	16	16	400	400	400	400	1.41
	2-1	40x15	25	25	600	590	600	600	5.02
	2-2	40x15	25	25	600	597	600	600	8.83
	2-3	40x15	25	25	600	600	600	600	5.96
	3-1	60x30	28	28	1800	1765	1800	1800	9.22
	3-2	60x30	29	29	1800	1755	1800	1796	6.16
	3-3	60x30	28	28	1800	1774	1800	1800	7.16
	4-1	60x60	49	49	3600	3528	3580	3591	16.36
	4-2	60x60	49	49	3600	3524	3564	3588	6.96
	4-3	60x60	49	49	3600	3544	3580	3594	15.98
	5-1	60x90	73	73	5400	5308	5342	5396	62.57
	5-2	60x90	73	73	5400	5313	5361	5400	69.53
	5-3	60x90	73	73	5400	5312	5375	5392	80.83
	6-1	80x120	97	97	9600	9470	9548	9582	150.05
	6-2	80x120	97	97	9600	9453	9448	9595	92.27
	6-3	80x120	97	97	9600	9450	9565	9582	129.01
	7-1	160x240	196	196	38400	37661	38026	38146	808.03
	7-2	160x240	197	197	38400	37939	38145	38374	357.85
	7-3	160x240	196	196	38400	37745	37867	38254	650.02
Mean Percentage deviation from optimum						1.68 %	0.42 %	0.17 %	

Table 8: Computational results - doubly constrained problems.

Source of problem	Inst.	Dim. (L x W)	m	M	UB	PH	GRASP	TABU	MPGA	CPU time (s)
Beasley (1985)	1	10x10	5	10	164	164	164	164	164	0.00
	2	10x10	7	17	230	225	225	225	225	0.00
	3	10x10	10	21	247	220	220	220	220	0.00
	4	15x10	5	7	268	268	268	268	268	0.01
	5	15x10	7	14	358	301	301	301	301	0.01
	6	15x10	10	15	289	265	252	265	265	0.08
	7	20x20	5	8	430	430	430	430	430	0.01
	8	20x20	7	13	834	819	819	819	819	0.01
	9	20x20	10	18	924	924	924	924	924	0.01
	10	30x30	5	13	n/f	n/f	n/f	n/f	n/f	0.02
	11	30x30	7	15	1688	1505	1518	1518	1518	0.47
	12	30x30	10	22	1865	1666	1648	1672	1672	0.05
Hadjiconstantinou and Christofides (1995)	3	30x30	7	7	1178	1178	1178	1178	1178	0.01
	11	30x30	15	15	1270	1216	1216	1216	1216	0.02
Wang (1983)		70x40	19	42	2726	2499	2700	2716	2716	0.03
Christofides and Whitlock (1977)	3	40x70	20	62	1860	1600	1720	1720	1720	0.03
Fekete and Schepers (1997a)	1	100x100	15	50	27718	25373	24869	25384	25384	16.87
	2	100x100	30	30	22502	17789	19083	19657	20678	1.22
	3	100x100	30	30	n/f	n/f	n/f	n/f	n/f	0.58
	4	100x100	33	61	32893	27556	27898	28974	28974	0.42
	5	100x100	29	97	27923	21997	22011	22011	22140	4.53
Mean Percentage deviation from upper bound						8.11	7.36	6.62	6.36	

5 Concluding remarks

In this paper we addressed a constrained two-dimensional (2D) packing problem, where a fixed set of small rectangles has to be cut from a larger stock rectangle so as to maximize the value of the rectangles packed. An algorithm which hybridizes a placement strategy with a multi-population genetic algorithm based on random keys was proposed. The approach was tested on four sets of instances taken from the literature and compared with other four approaches. The experimental results demonstrate the effectiveness and robustness of the proposed heuristic when compared with other approaches.

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