AN ANNOTATED BIBLIOGRAPHY OF NETWORK INTERIOR POINT METHODS

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Abstract. This paper presents an annotated bibliography on interior point methods for solving network flow problems. We consider single and multi-commodity network flow problems, as well as preconditioners used in implementations of conjugate gradient methods for solving the normal systems of equations that arise in interior network flow algorithms. Applications in electrical engineering and miscellaneous papers complete the bibliography. The collection includes papers published in journals, books, Ph.D. dissertations, and unpublished technical reports.

1. Introduction

Many problems arising in transportation, communications, and manufacturing can be modeled as network flow problems (Ahuja et al. (1993)). In these problems, one seeks an optimal way to move flow, such as overnight mail, information, buses, or electrical currents, on a network, such as a postal network, a computer network, a transportation grid, or a power grid. Among these optimization problems, many are special classes of linear programming problems, with combinatorial properties that enable development of efficient solution techniques.

The focus of this annotated bibliography is on recent computational approaches, based on interior point methods, for solving large scale network flow problems. In the last two decades, many other approaches have been developed. A history of computational approaches up to 1977 is summarized in Bradley et al. (1977). Several computational studies established the fact that specialized network simplex algorithms are orders of magnitude faster than the best linear programming codes of that time. (See, e.g. the studies in Glover and Klingman (1981); Grigoriadis (1986); Kennington and Helgason (1980) and Mulvey (1978)).

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In 1984, N. Karmarkar introduced a new polynomial time algorithm for solving linear programming problems (Karmarkar (1984)). This algorithm and many of its variants, known as interior point methods, have been used to efficiently solve a wide range of mathematical programming problems, including network flow problems.

The annotated bibliography is organized as follows. Section 2 deals with single-commodity minimum cost network flow. Most interior point network flow implementations compute the search direction approximately, using a preconditioned conjugate gradient algorithm. Section 3 covers work on preconditioners for network flow. In Section 4, papers on multi-commodity interior point network flow algorithms are examined. The electrical engineering field has seen the most applications of network interior point methods in the literature. Papers on this application area are covered in Section 5. Finally, papers which could not be classified into the above categories are grouped together in Section 6.

We made use of several world wide web resources to compile this list of papers. Our search focussed mainly on, but was not limited to, Google (`google.com`), ISI Web of Science (`isiknowledge.com`), and CiteSeer (`citeseer.nj.nec.com`), where we searched for papers that cited work on network interior point methods previously known to us, as well as papers containing the terms *interior point* and *network*.

R.K. Ahuja, T.L. Magnanti, and J.B. Orlin. *Network Flows.* Prentice Hall, Englewood Cliffs, NJ, 1993.

G.H. Bradley, G.G. Brown, and G.W. Graves. Design and implementation of large scale primal transshipment algorithms. *Management Science*, 24:1–34, 1977.

F. Glover and D. Klingman. The simplex SON algorithm for LP/embedded network problems. *Mathematical Programming Study*, 15:148–176, 1981.

M.D. Grigoriadis. An efficient implementation of the network simplex method. *Mathematical Programming Study*, 26:83–111, 1986.

J.L. Kennington and R.V. Helgason. *Algorithms for network programming.* John Wiley and Sons, New York, NY, 1980.

J.M. Mulvey. Testing of a large scale network optimization program. *Mathematical Programming*, 15:291–315, 1978.

N. Karmarkar. A new polynomial-time algorithm for linear programming. *Combinatorica*, 4:373–395, 1984.

> Narendra Karmarkar's seminal paper on a polynomial-time interior point algorithm for linear programming.

2. Single-commodity network flow

In the mid-1980s, evidence began to surface that interior point methods could solve some classes of linear programming problems more efficiently than the simplex method. Much research was done on implementation issues of interior point methods and many efficient implementations resulted. Following the steps of past research on the network simplex method, groups of researchers began investigating the applicability of interior point methods to solve network flow problems.

Several papers report on computational studies comparing interior point algorithms for general linear programming with specialized algorithms for network flows. Others exploit the special structure of network flow problems to speedup the computation of interior point methods. The bulk of the work investigates how to apply preconditioned conjugate gradient methods for inexact computation of the search direction. This resulted in a number of efficient preconditioners and techniques to determine the optimal solution from the interior of the polytope.

J. Aronson, R. Barr, R. Helgason, J. Kennington, A. Loh, and H. Zaki. The projective transformation algorithm of Karmarkar: A computational experiment with assignment problems. Technical Report 85-OR-3, Department of Operations Research, Southern Methodist University, Dallas, TX, August 1985.

> PTANET, a variant of Karmarkar's projective transformation algorithm for solving network flow problems is described. The code, an implementation of a primal projective algorithm, used the LSQR subroutine of Paige and Saunders (1982) to compute the search direction, exploiting the network structure. No preconditioner was used, possibly explaining the large number of LSQR iterations taken. Computational testing was limited to small dense assignment problems. PTANET was compared with NETFLO and IBM MPSX/370 and was found to be

hundreds of times slower than NETFLO and tens of times slower than IBM MPSX/370.

I. Adler, N. Karmarkar, M.G.C. Resende, and G. Veiga. An implementation of Karmarkar's algorithm for linear programming. *Mathematical Programming*, 44: 297–335, 1989a.

This paper presents and tests an implementation of the dual affine scaling algorithm with tests on assignment and multicommodity network flow problems. The use of preconditioned conjugate gradient method is described.

I. Adler, N. Karmarkar, M.G.C. Resende, and G. Veiga. Data structures and programming techniques for the implementation of Karmarkar's algorithm. *ORSA Journal on Computing*, 1:84–106, 1989b.

Describes implementation details for Adler et al. (1989a), including the use of the preconditioned conjugate gradient method.

Y. C. Cheng, D. J. Houck, J. M. Liu, M. S. Meketon, L. Slutsman, R. J. Vanderbei, and P. Wang. The AT&T KORBX System. *AT&T Technical Journal*, 68:7–19, 1989.

This paper compares the dual affine scaling implementation in AT&T's KORBX system (which uses direct factorization) with Bertsekas' relaxation algorithm RELAXT-3 on a set of NETGEN test problems. RELAXT-3 was up to two orders of magnitude faster than KORBX.

A. Rajan. An empirical comparison of KORBX against RELAXT, a special code for network flow problems. Technical report, AT&T Bell Laboratories, Holmdel, NJ, 1989.

A computational study is presented comparing the dual affine scaling implementation in AT&T's KORBX mathematical programming system (1989) with the implementation of Bertsekas' relaxation algorithm, RELAXT-3 (1988), on a set of problems generated with NETGEN. RELAXT-3 was up to two orders of magnitude faster than KORBX. The KORBX implementation solved the search direction system using direct factorization and takes no advantage of the structure of the network.

Q.-J. Yeh. *A reduced dual affine scaling algorithm for solving assignment and transportation problems*. PhD thesis, Columbia University, New York, NY, 1989.

Proposed and implemented the reduced dual affine scaling algorithm for solving assignment and transportation problems. The code uses a diagonal preconditioner and switches to the simplex method at the end.

A. Armacost and S. Mehrotra. Computational comparison of the network simplex method with the affine scaling method. *Opsearch*, 28:26–43, 1991.

The authors compare their implementations of a network simplex method and the dual affine scaling algorithm using direct factorization. Results show simplex is faster but difference diminishes with the increase of instance sizes.

N.K. Karmarkar and K.G. Ramakrishnan. Computational results of an interior point algorithm for large scale linear programming. *Mathematical Programming*, 52:555–586, 1991.

A dual affine scaling algorithm with centering which uses a preconditioned conjugate gradient method is described and tested on a wide class of maximum flow and minimum cost flow problems. Comparisons are made with the general linear programming code in MINOS, with speedups of up to 252.

C. Wallacher and U. Zimmermann. A combinatorial interior point method for network flow problems. *Mathematical Programming*, 56(3):321–335, 1992.

A combinatorial interior point method for solving minimum cost flow problems, based on a variant of the algorithm of Karmarkar described in Gonzaga (1990), is presented. This algorithm exploits the special combinatorial structure of networks to evaluate the search direction.

I.C. Choi and D. Goldfarb. Exploiting special structure in a primal dual path-following algorithm. *Mathematical Programming*, 58(1):33–52, 1993.

A primal-dual path-following algorithm that explicitly handles upper bounds, generalized upper bounds, variable upper bounds, and block diagonal structure is presented. It shows how the structure of time-staged problems and network flow problems can be exploited.

A. Joshi, A.S. Goldstein, and P.M. Vaidya. A fast implementation of a path-following algorithm for maximizing a linear function over a network polytope. In D.S. Johnson and C.C. McGeoch, editors, *Network Flows and Matching: First DIMACS Implementation Challenge*, volume 12 of *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, pages 267–298. American Mathematical Society, 1993.

The implementation and testing of a primal path following interior point algorithm using a preconditioned conjugate gradient with the maximum weighted spanning tree preconditioner to compute the search direction is described. Though the performance of this code, relative to NETLFO and RELAXT-3, was not competitive on most instances, it was shown to improve with problem size. The largest speedups with respect to NETFLO and RELAXT-3 were 2.4 and 25, respectively.

J.A. Kaliski and Y.Y. Ye. A decomposition variant of the potential reduction algorithm for linear programming. *Management Science*, 39:757–776, 1993.

The implementation of a build-up variant of the dual potential reduction algorithm for solving transportation problems with many more variables than constraints is described. The implementation uses the preconditioned conjugate gradient with the maximum weighted spanning tree preconditioner to compute the search direction. They differ from previous implementations in the way they compute the spanning tree, using a hybrid QuickSort procedure. To speed up the convergence of the duality gap, they use an indicator function to guess a dual face, and project the dual interior solution on the face and test optimality. On small problems, with 50 to 2000 nodes and arcs, NETFLO was 2 to 10 times faster than the build-up variant of the dual potential reduction algorithm.

K.G. Ramakrishnan, N.K. Karmarkar, and A.P. Kamath. An approximate dual projective algorithm for solving assignment problems. In D.S. Johnson and C.C. McGeoch, editors, *Network Flows and Matching: First DIMACS Implementation Challenge*, volume 12 of *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, pages 431–451. American Mathematical Society, 1993.

This paper extends the approximate dual projective (ADP) code by adding an initial solution module and a stopping heuristic module for handling assignment problems. The code, called ADP/A, was used to solve assignment problems having up to 32,768 nodes. That code was also used to solve large dense assignment problems to verify conjectures on the value of the optimal assignments of randomly distributed cost data. Computational results on dense assignment problems with up to 20,000 nodes are reported.

M.G.C. Resende and G. Veiga. An efficient implementation of a network interior point method. In D.S. Johnson and C.C. McGeoch, editors, *Network Flows and Matching: First DIMACS Implementation Challenge*, volume 12 of *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, pages 299–348. American Mathematical Society, 1993a.

DLNET, an implementation of the dual affine scaling algorithm for minimum cost capacitated network flow problems is described and tested. The efficiency of this implementation is the result of three factors: the small number of iterations taken by interior point methods, efficient solution of the linear system that determines the ascent direction using a preconditioned conjugate gradient algorithm and strategies to produce an optimal primal integer solution.

M.G.C. Resende and G. Veiga. An implementation of the dual affine scaling algorithm for minimum cost flow on bipartite uncapaciated networks. *SIAM Journal on Optimization*, 3:516–537, 1993b.

Compares a dual affine scaling algorithm which uses a parallel diagonally preconditioned conjugate gradient algorithm with the network simplex code NETFLO and Bertsekas' relaxation algorithm RELAXT-3 on linear assignment problems. The interior point algorithm is shown to be the fastest on large instances.

M.G.C. Resende and G. Veiga. Computing the projection in an interior point algorithm: An experimental comparison. *Investigación Operativa*, 3:81–92, 1993c.

On large assignment problems, a speedup factor over 116 is shown for a dual affine scaling method using a preconditioned conjugate gradient method compared to another based on direct factorization.

M.G.C. Resende, T. Tsuchiya, and G. Veiga. Identifying the optimal face of a network linear program with a globally convergent interior point method. In W.W. Hager, D.W. Hearn, and P.M. Pardalos, editors, *Large scale optimization: State of the art*, pages 362–387. Kluwer Academic Publishers, 1994.

Strategies to identify the optimal face of a minimum cost network flow problem are studied. In the computational experiments described, one of the proposed optimality indicators is used to implement an early stopping criterion in DLNET, an implementation of the dual affine scaling algorithm for solving minimum cost network flow problems. The experiments suggest that the new indicator is far more robust than the one used in earlier versions of DLNET.

A. Joshi. *Topics in optimization and sparse linear systems.* PhD thesis, University of Illinois, Urbana-Champaign, IL, 1996.

Among other problems, network optimization is studied. The quality of a new graph-based preconditioner for large sparse Symmetric Positive Definite Diagonally Dominant (SPDDD) linear systems is analyzed.

L. Portugal, F. Bastos, J. Júdice, J. Paixão, and T. Terlaky. An investigation of interior point algorithms for the linear transportation problem. *SIAM J. Sci. Computing*, 17:1202–1223, 1996.

A preconditioner based on an incomplete QR decomposition is used to implement dual affine scaling, primal-dual, and predictor-corrector interior point methods to solve transportation problems.

G. Veiga. *Sur l'implatation des méthodes de points intérieurs por la programmation linéaire.* PhD thesis, Université Paris 13, Institut Galilée, Paris, France, 1997.

Describes implementations of interior point methods for general linear programming as well as network programming.

L. Portugal, M.G.C. Resende, G. Veiga, and J. Júdice. A truncated primal-infeasible dual-feasible network interior point method. *Networks*, 35:91–108, 2000.

The truncated primal-infeasible dual-feasible interior point algorithm for linear programming is introduced and an implementation of this

algorithm for solving the minimum cost network flow problem is described. In each iteration, the linear system that determines the search direction is computed inexactly, and the norm of the resulting residual vector is used in the stopping criteria of the iterative solver employed for the solution of the system. In the implementation, a preconditioned conjugate gradient method is used as the iterative solver. The code (PDNET) is tested on a large set of standard minimum cost network flow test problems. Computational results indicate that the implementation is competitive with state-of-the-art network flow codes.

D. Goldfarb and Y. Lin. Combinatorial interior point methods for generalized network flow problems. *Mathematical Programming*, 93:227–246, 2002.

Combinatorial interior point methods for the generalized minimum cost flow and the generalized circulation problems are proposed. The algorithms are based on the combinatorial interior point method for the minimum cost network flow problem of Wallacher and Zimmermann. They maintain interior point solutions and reduce the value of a potential function. Flow is augmented along a generalized circulation, which is computed by solving a two variables per inequality system.

3. Preconditioners for conjugate gradient method

The computational efficiency of interior point network flow methods relies heavily on a preconditioned conjugate gradient algorithm to solve the direction finding system at each iteration. The preconditioned conjugate gradient algorithm is used to solve

$$M^{-1}(AD_kA^\top)\Delta y = M^{-1}\bar{b}$$

for $\Delta y \in \mathbb{R}^m$, where $A \in \mathbb{R}^{m\times n}$ is the node arc incidence matrix and $M \in \mathbb{R}^{m\times m}$ is a positive definite matrix. The vector $\bar{b} \in \mathbb{R}^m$ and the diagonal matrix $D_k \in \mathbb{R}^{n\times n}$ depend on the interior point algorithm used. For example, in the case of the dual affine scaling algorithm, $\bar{b} = b - AZ_k^2 D_k^u$, and $D_k = (Z_k^2 + S_k^2)^{-1}$, where Z_k^2 and S_k^2 are diagonal matrices. The objective is to make the preconditioned matrix

$$M^{-1}(AD_kA^\top)$$

less ill-conditioned than $AD_kA^\top$, and improve the convergence of the conjugate gradient algorithm.

Some of the references in Section 2 also cover preconditioners.

P.M. Vaidya. Solving linear equations with symmetric diagonally dominant matrices by constructing good preconditioners. Technical report, Department of Computer Science, University of Illinois at Urbana-Champaign, Urbana, IL, 1990.

A new class of preconditioners and a new technique for analyzing preconditioners is presented. Several families of preconditioners are proposed. The simplest one is based on maximum spanning trees (MST) of the underlying graph of the matrix. The second one augments the MST with extra edges to speed up convergence. A third one is based on a maximum-weight basis (MWB) of the matroid associated with the graph of the matrix.

S. Mehrotra and J. Wang. Conjugate gradient based implementation of interior point methods for network flow problems. In L. Adams and L. Nazareth, editors, *Linear and Nonlinear Conjugate Gradient-Related Methods*, pages 124–142. SIAM, 1996.

A diagonally compensated maximum spanning tree preconditioner for network interior point methods is proposed and tested.

V. Baryamureeba and T. Steihaug. On the convergence of an inexact primal-dual interior point method for linear programming. Technical Report 188, Department of Informatics, University of Bergen, 2000.

The inexact primal-dual interior point method of Kojima, Megiddo, and Mizuno which is discussed in this paper chooses a new iterate along an approximation to the Newton direction. The inexact variation takes distinct step length in both the primal and dual spaces and is globally convergent.

A. Frangioni and C. Gentile. New preconditioners for KKT systems of network flow problems. Technical Report 539, Intituto di Analisi dei Sistemi ed Informatica, 2000.

New preconditioners for network interior point methods are proposed. The preconditioners are based on extracting a proper subgraph of the original graph containing a spanning tree.

W. Wang and D.P. O'Leary. Adaptive use of iterative methods in predictor-corrector interior point methods for linear programming. *Numerical Algorithms*, 25(1–4):387–406, 2000.

An adaptive algorithm that changes strategy over the course of the interior point algorithm is developed. The algorithm determines dynamically whether the preconditioner should be held constant, updated, or recomputed, switching to a direct method when it predicts that an iterative method will be too expensive.

A. Frangioni and S.S. Capizzano. Spectral analysis of (sequences of) graph matrices *SIAM Journal on Matrix Analysis and Applications*, 23(2):339–348, 2001.

Graph matrices with extreme singular values are studied, obtaining asymptotically tight lower and upper bounds. This analysis is used to obtain estimates on the spectral condition number of some weighted graph matrices. Possible preconditioning strategies within interior-point methods for network flow problems are discussed.

W.C. Wang. The convergence of an interior-point method using modified search directions in final iterations. *Computers & Mathematics with Applications*, 44 (3–4):347–356, 2002.

An asymptotic analysis of a primal-dual algorithm for linear programming that uses modified search directions in the final iterations is provided. The algorithm determines the search directions by solving the normal equations using the preconditioned conjugate gradient algorithm. The analysis proves the global convergence of this interior-point method.

J.J. Júdice, J.M. Patrício, L.F. Portugal, M.G.C. Resende, and G. Veiga. A study of preconditioners for network interior point methods. *Computational Optimization and Applications*, 24:5–35, 2003a.

Several preconditioners available for network interior point methods are studied and compared. Upper bounds for the condition number of the preconditioned matrices are derived. A computational comparison using a set of standard problems improves the understanding of the effectiveness of preconditioners in network interior point methods.

R.D.C. Monteiro, J. W. O'Neal, and T. Tsuchiya. Uniform boundedness of a preconditioned normal matrix used in interior point methods. Technical report, School of Industrial Engineering, Georgia Institute of Technology, 2003.

A proof is given that the maximum weighted spanning tree preconditioner for the normal system of equations that arises in network interior point methods produces scaled matrices with uniformly bounded condition numbers as the scaling matrix varies over the set of all positive diagonal matrices. The condition number is bounded above by $m(n-m+1)$, where m is the number of nodes and n is the number of acs.

J.J. Júdice, J.M. Patrício, L.F. Portugal, M.G.C. Resende, and G. Veiga. ERRATA: A study of preconditioners for network interior point methods. Technical report, AT&T Labs Research, Florham Park, NJ 07932 USA, 2003b.

This paper corrects an error in a proof of a corrolary in Júdice et al. (2003a).

4. Multi-commodity network flow

In many practical applications, flows in networks with more than one commodity need to be optimized. In the multi-commodity network flow problem, k commodities are to be moved in the network. Let x_{ij}^k denote the flow of commodity k in arc (i,j). The multi-commodity network flow problem can be formulated as the following linear program:

$$\min \sum_{k \in \mathcal{K}} \sum_{(i,j) \in \mathcal{A}} c_{ij}^k x_{ij}^k$$

subject to:

$$\sum_{(j,l) \in \mathcal{A}} x_{jl}^k - \sum_{(l,j) \in \mathcal{A}} x_{lj}^k = b_j^k, \;\; j \in \mathcal{N}, \;\; k \in \mathcal{K}$$

$$\sum_{k \in \mathcal{K}} x_{ij}^k \le u_{ij}, \;\; (i,j) \in \mathcal{A},$$

$$x_{ij}^k \ge 0, \;\; (i,j) \in \mathcal{A}, \;\; k \in \mathcal{K},$$

where $\mathcal{N}$ denotes the set of nodes, $\mathcal{A}$ denotes the set of arcs, and $\mathcal{K}$ denotes the set of commodities. The minimum cost network flow problem is a special case of the multi-commodity network flow problem, in which there is only one commodity.

One of the first commercial applications of interior point methods (circa 1987) was solving U.S. Military Airlift Command (MAC, now called Air Mobility Command, AMC) multi-commodity flow problems with KORBX, AT&T's interior point LP solver. The patient-distribution system (PDS) model is used by AMC to plan patient evacuation from a war theater.

Not only general-purpose linear programming solvers have been applied to solve multi-commodity flow problems. Interior point linear programming algorithms have been specialized to exploit the special structure of the problem. These specialized algorithms have been able to solve instances with up to 260,000 constraints and 2.5 million variables.

Regarding complexity of multi-commodity flow problems, the best known algorithm is the interior point method analyzed by Kamath and Palmon (1995). One variant of this interior point algorithm finds an exact solution in $O(k^{2.5}n^{1.5}mL \log(nDU))$, where $L = km + n$, D is the largest demand, U is the largest capacity, and k is the number of commodities. This variant uses the conjugate gradient method. The second variant, which has complexity bound of $O((k^{0.5}n^3 + km^{1.5}n^{1.5})L \log(mDU))$, can be improved to $O((k^{0.5}n^{2.7} + km^{1.2}n^{1.5} + kn^{2.5})L \log(mDU))$, using fast matrix multiplication. This variant uses matrix inversion and rank one updates.

Previously, the fastest known algorithm for multi-commodity flow was also an interior point algorithm, due to Kapoor and Vaidya (1986) and Vaidya (1989). This algorithm achieves a complexity bound of $O(k^3n^{0.5}m^3L \log(mDU))$ and reduces to $O(k^{2.5}n^{0.5}m^2L \log(mDU))$ using fast matrix multiplication. It should be noted, however, that algorithms using fast matrix multiplication are not of practical interest, unless the dimensions are very large.

S. Kapoor and P.M. Vaidya. Fast algorithms for convex quadratic programming and multicommodity flows. In *Proceedings of 18th Annual ACM Symposium on the Theory of Computing*, pages 147–159, 1986.

In the first part of the paper, Karmarkar's interior point method is extended to give an algorithm for convex quadratic programming. In the second part, the linear program describing the multi-commodity flow problem is solved.

P.M. Vaidya. Speeding up linear programming using fast matrix multiplication. In *Proceedings of 30th IEEE Annual Symposium on Foundations of Computer Science*, pages 332–337, 1989.

This paper shows how to speed up interior point algorithms for linear programming by using fast matrix multiplication.

W. Carolan, J. Hill, J. Kennington, S. Niemi, and S. Wichman. An empirical evaluation of the KORBX algorithms for military airlift applications. *Operations Research*, 38(2):240–248, 1990.

This paper reports on the use of the AT&T KORBX linear programming system installed at the U.S. Military Airlift Command at Scott Air Force Base to solve a set of linear programming applications, among them multi-commodity flow problems.

A.P. Kamath, N.K. Karmarkar, and K.G. Ramakrishnan. Computational and complexity results for an interior point algorithm for multicommodity flow problems. Technical Report TR-21/93, Dipartimento di Informatica, Università di Pisa, 1993.

Computational and complexity results for the Approximate Dual Projective (ADP) interior point algorithm for multi-commodity flow problems are presented. Computational results on problems with up to 169,728 rows and 657,152 columns are reported.

A.P. Kamath and O. Palmon. Improved interior point algorithms for exact and approximate solution of multicommodity flow problems. In *Proceedings of the Sixth Annual ACM–SIAM Symposium on Discrete Algorithms*, pages 502–511, January 1995.

A new interior-point quadratic-optimization based polynomial algorithm for the multi-commodity flow problem and its variants are presented. This algorithm improves running time in the approximate case by a polynomial factor. For many cases, the exact bounds are better as well. The algorithm exploits the underlying structure of the problem to solve it efficiently.

S. Kapoor and P.M. Vaidya. Speeding up Karmarkar's algorithm for multicommodity flows. *Mathematical Programming*, 73(1):111–127, 1996.

Karmarkar's linear programming algorithm is sped up for the case of multi-commodity flows by exploiting the special structure of the constraint matrix.

I. Lustig and E. Rothberg. Gigaflops in linear programming. *Operations Research Letters*, 18(4):157–165, 1996.

This paper describes a parallel implementation of the CPLEX barrier interior point algorithm on a SGI Power Challenge with 16 processors. The authors report the solution of the multi-commodity flow patient distribution system problem PDS-20 in 409 seconds using all 16 processors.

E. Yamakawa, Y. Matsubara, and M. Fukushima. A parallel primal-dual interior point method for multicommodity flow problems with quadratic costs. *Journal of the Operations Research Society of Japan*, 39(4):566–591, 1996.

A parallel implementation of a primal-dual interior point method for multi-commodity flow problems with separable quadratic costs is described. The implementation exploits the block structure of the problem and uses the conjugate gradient method with an appropriate preconditioner to determine the search direction. Computational results indicate that the proposed approach is efficient for large-scale multi-commodity flow problems.

J.J. Júdice, L.F. Portugal, M.G.C. Resende, and G. Veiga. A truncated interior point method for the solution of minimum cost flow problems on an undirected multicommodity network. In *Proceedings of First Portuguese National Telecommunications Conference*, pages 381–384, 1997.

Several preconditioners for single-commodity network interior point methods are extended to handle multi-commodity network flow problems. In Portuguese.

J. Castro. A specialized interior-point algorithm for multicommodity network flows. *SIAM Journal on Optimization*, 10(3):852–877, 2000.

A new specialized interior-point algorithm for multi-commodity flows uses both a preconditioned conjugate gradient solver and a sparse Cholesky factorization to solve a linear system of equations at each iteration of the algorithm. The preconditioner exploits the structure of the problem. An implementation of the algorithm is compared to state-of-the-art packages for multi-commodity flows on instances with up to 700,000 variables and 150,000 constraints.

J. Castro and A. Frangioni. A parallel implementation of an interior-point algorithm for multicommodity network flows. In *Vector and Parallel Processing – VECPAR 2000*, volume 1981 of *Lecture Notes in Computer Science*, pages 301–315. Springer-Verlag, 2001.

A coarse-grained parallel implementation of the specialized interior-point algorithm for multi-commodity network flows introduced in Castro (2000) is presented. Computational results are described on problems with up to 2.5 million variables and 260,000 constraints, indicating that the method is competitive on large, difficult multi-commodity flow problems.

R. Chardaire and A. Lisser. Simplex and interior point specialized algorithms for solving nonoriented multicommodity flow problems. *Operations Research*, 50: 260–276, 2002.

This paper presents various approaches based on specializations of the simplex algorithm and interior-point methods to solve nonoriented multi-commodity flow problems.

5. Applications in electrical engineering

Interior point methods have been applied to power systems problems in electrical engineering since the early 1990s. This includes problems related to optimal power flow, transfer capability assessment, economic dispatching, spinning reserve allocation, and network constrained security control. They have been solved with both direct factorization and preconditioned conjugate gradient based interior point methods.

K. Ponnambalam, V.H. Quintana, and A. Vannelli. A fast algorithm for power-system optimization problems using an interior point method. *IEEE Transactions on Power Systems*, 7(2):892–899, 1992.

An implementation of the dual affine scaling algorithm is described in detail and some computational results are presented. The normal equations are solved using a preconditioned conjugate gradient method.

C.N. Lu and M.R. Unum. Network constrained security control using an interior-point algorithm. *IEEE Transactions on Power Systems*, 8(3):1068–1076, 1993.

A preliminary implementation of an interior point algorithm is described and test results on the network constrained security control problem are presented.

J.C.O. Mello, A.C.G. Melo, and S. Granville. Simultaneous transfer capability assessment by combining interior point methods and Monte Carlo simulation. *IEEE Transactions on Power Systems*, 12(2):736–742, 1997.

A methodology to evaluate the maximum simultaneous power transfer of large interconnected power systems is described. The approach combines Monte Carlo simulation and AC optimal power flow, solved by a direct interior point algorithm.

G. Irisarri, L.M. Kimball, K.A. Clements, A. Bagchi, and P.W. Davis. Economic dispatch with network and ramping constraints via interior point methods. *IEEE Transactions on Power Systems*, 13(1):236–242, 1998.

An approach to the economic dispatch problem that combines both time-separated constraints (e.g., demand and network flow) and inter-temporal constraints (e.g., ramping) into a single optimization problem that can be solved efficiently by interior point methods is described.

A. Garzillo, M. Innorta, and M. Ricci. The flexibility of interior point based optimal power flow algorithms facing critical network situations. *International Journal of Electrical Power & Energy Systems*, 21(8):579–584, 1999.

Extensions of the basic optimal power flow model for the analysis of problems concerning the installations of reactive power shunt compensators and the optimization of load shedding are described. Interior point procedures are tested on several real networks and shown to be more accurate than the traditional de-coupled programs of optimal power flow.

E. Rezania and S.M. Shahidehpour. Calculation of transfer capability of power systems using an efficient predictor-corrector primal dual interior point algorithm. *Electric Machines and Power Systems*, 27(1):25–37, 1999.

A new algorithm using the primal-dual interior point method with the predictor-corrector for determination of maximum loadability in a power system is presented in this paper.

R.A. Jabr, A.H. Coonick, and B.J. Cory. A study of the homogeneous algorithm for dynamic economic dispatch with network constraints and transmission losses. *IEEE Transactions on Power Systems*, 15(2):605–611, 2000.

This paper presents a study of the homogeneous interior point method for the economic dispatch problem that combines both independent blocks of constraints (generation demand balance, network flows) and coupling constraints (ramping) into a single optimization problem.

K. Xie and Y.H. Song. Optimal spinning reserve allocation with full AC network constraints via a nonlinear interior point method. *Electric Machines and Power Systems*, 28(11):1071–1090, 2000.

A primal-dual interior point method, efficiently handling both equality constraints and inequality constraints, is employed to solve a dynamic optimal fouler flow problem.

E. Rezania and S.M. Shahidehpour. Real power loss minimization using interior point method. *International Journal of Electrical Power & Energy Systems*, 23 (1):45–56, 2001.

A predictor-corrector primal-dual interior point method is applied to reactive power optimization. Matrix inversion computation as well as second-order derivatives of Hessian matrix are avoided by introducing a linear model in which control variables and voltage increments are linked by a modified Jacobian matrix and transmission losses are represented as a function of voltage increments.

K. Xie and Y.H. Song. Dynamic optimal power flow by interior point methods. In *IEE Proceedings – Generation, Transmission, and Distribution*, volume 148, pages 76–84, 2001a.

An algorithm based on nonlinear interior point methods is developed for the dynamic optimal power flow problem in which both time-separated and time-related constraints are considered and solved as a single optimization problem. Numerical results on systems having 30 to 118 busbars, with up to 24-hour period, are employed to illustrate efficiency and robustness of the method.

K. Xie and Y.H. Song. Power market oriented optimal power flow via an interior point method. In *IEE Proceedings – Generation, Transmission, and Distribution*, volume 148, pages 549–556, 2001b.

A nonlinear interior point method based optimal power flow (OPF) is proposed for power market oriented OPF, inheriting both the supersparsity technique of the Newton OPF and the advantages of interior point methods in handling inequality constraints efficiently by slightly modifying Hessian matrix entries without changing their sparsity structure.

L.C.A. Ferreira, A.C.Z. de Souza, S. Granville, and J.W.M. Lima. Interior point method applied to voltage collapse problems and system-losses-reduction. In *IEE Proceedings – Generation, Transmission, and Distribution*, volume 149, pages 165–170, 2002.

An optimization technique based on interior points is used to improve the system-operating conditions. Tests are carried out using the interconnected Southeastern Brazilian system, where all the reactive power limits are taken into account.

R.A. Jabr, A.H. Coonick, and B.J. Cory. A primal-dual interior point method for optimal power flow dispatching. *IEEE Transactions on Power Systems*, 17(3): 654–662, 2002.

The solution of the optimal power flow dispatching problem by a primal-dual interior point method is considered.

L.M. Kimball, K.A. Clements, P.W. Davis, and I. Nejdawi. Multiperiod hydrothermal economic dispatch by an interior point method. *Mathematical Problems in Engineering*, 8:33–42, 2002.

This paper presents an interior point algorithm to solve the multiperiod hydrothermal economic dispatch problem.

6. Miscellaneous

We conclude this annotated bibliography with a few papers which we could not classify in the categories addressed in Sections 2–5.

P.M. Pardalos and K.G. Ramakrishnan. On the expected value of random assignment problems: Experimental results and open questions. *Computational Optimization and Applications*, 3:261–271, 1993.

A dual affine scaling interior point algorithm with centering is used to solve large instances of linear assignment problems to verify a conjecture on the expected value of random assignment problems.

L.K. Grover. Fast interior-point methods for bipartite matching. *SIAM Journal on Optimization*, 5(4):740–769, 1995.

By using path following interior point methods, it is shown that it is possible to derive the optimal bipartite matching in $O(v^{1/2})$ time, where v is the number of vertices in the graph, thus improving upon the results of Vaidya (1990) and Goldberg, Plotkin, and Vaidya (1990).

M.G.C. Resende and P.M. Pardalos. Interior point algorithms for network flow problems. In J.E. Beasley, editor, *Advances in linear and integer programming*, pages 147–187. Oxford University Press, 1996.

This chapter reviews interior point methods for network flow problems up to 1996.

G.L. Xue, T.P. Lillys, and Dougherty D, E. Computing the minimum cost pipe network interconnecting one sink and many sources. *SIAM Journal on Optimization*, 10(1):22–42, 1999.

The problem of computing the minimum cost pipe network interconnecting a given set of wells and a treatment site is studied. An interior-point algorithm is used to find the a minimum cost pipe network.

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